Uncertainty Theory

Fifth Edition

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Preface

When no samples are available to estimate a probability distribution, we have to invite some domain experts to evaluate the belief degree that each event will happen. Perhaps some people think that the belief degree should be modeled by subjective probability or fuzzy set theory. However, it is usually inappropriate because both of them may lead to counterintuitive results in this case. In order to rationally deal with personal belief degrees, uncertainty theory was founded in 2007 and subsequently studied by many researchers. Nowadays, uncertainty theory has become a branch of mathematics.

Uncertain Measure

The most fundamental concept is uncertain measure that is a type of set function satisfying the axioms of uncertainty theory. It is used to indicate the belief degree that an uncertain event may happen. Chapter 1 will introduce normality, duality, subadditivity and product axioms. From those four axioms, this chapter will also present uncertain measure, product uncertain measure, and conditional uncertain measure.

Uncertain Variable

Uncertain variable is a measurable function from an uncertainty space to the set of real numbers. It is used to represent quantities with uncertainty. Chapter 2 is devoted to uncertain variable, uncertainty distribution, independence, operational law, expected value, variance, moments, distance, entropy, conditional uncertainty distribution, uncertain sequence, uncertain vector, and uncertain matrix.

Uncertain Programming

Uncertain programming is a type of mathematical programming involving uncertain variables. Chapter 3 will provide a type of uncertain programming model with applications to machine scheduling problem, vehicle routing problem, and project scheduling problem. In addition, uncertain multiobjective programming, uncertain goal programming and uncertain multilevel programming are also documented.

Uncertain Risk Analysis

The term risk has been used in different ways in literature. In this book the risk is defined as the accidental loss plus the uncertain measure of such loss, and a risk index is defined as the uncertain measure that some specified loss occurs. Chapter 4 will introduce uncertain risk analysis that is a tool to quantify risk via uncertainty theory. As applications of uncertain risk analysis, Chapter 4 will also discuss structural risk analysis and investment risk analysis.

Uncertain Reliability Analysis

Reliability index is defined as the uncertain measure that some system is working. Chapter 5 will introduce uncertain reliability analysis that is a tool to deal with system reliability via uncertainty theory.

Uncertain Propositional Logic

Uncertain propositional logic is a generalization of propositional logic in which every proposition is abstracted into a Boolean uncertain variable and the truth value is defined as the uncertain measure that the proposition is true. Chapter 6 will present uncertain propositional logic and uncertain predicate logic. In addition, uncertain entailment is a methodology for determining the truth value of an uncertain proposition via the maximum uncertainty principle when the truth values of other uncertain propositions are given. Chapter 7 will discuss an uncertain entailment model from which uncertain modus ponens, uncertain modus tollens and uncertain hypothetical syllogism are deduced.

Uncertain Set

Uncertain set is a set-valued function on an uncertainty space, and attempts to model unsharp concepts like "young", "tall", "warm", and "most". The main difference between uncertain set and uncertain variable is that the former takes values of set and the latter takes values of point. Uncertain set theory will be introduced in Chapter 8.

Uncertain Logic

Some knowledge in human brain is actually an uncertain set. This fact encourages us to design an uncertain logic that is a methodology for calculating the truth values of uncertain propositions via uncertain set theory. Uncertain logic may provide a flexible means for extracting linguistic summary from a collection of raw data. Chapter 9 will be devoted to uncertain logic and linguistic summarizer.

Uncertain Inference

Uncertain inference is a process of deriving consequences from human knowledge via uncertain set theory. Chapter 10 will present a set of uncertain inference rules, uncertain system, and uncertain control with application to an inverted pendulum system.

Uncertain Process

An uncertain process is essentially a sequence of uncertain variables indexed by time. Thus an uncertain process is usually used to model uncertain phenomena that vary with time. Chapter 11 is devoted to basic concepts of uncertain process and uncertainty distribution. In addition, extreme value theorem, first hitting time and time integral of uncertain processes are also introduced. Chapter 12 deals with uncertain renewal process, renewal reward process, and alternating renewal process. Chapter 12 also provides block replacement policy, age replacement policy, and an uncertain insurance model.

Uncertain Calculus

Uncertain calculus is a branch of mathematics that deals with differentiation and integration of uncertain processes. Chapter 13 will introduce Liu process that is a stationary independent increment process whose increments are normal uncertain variables, and discuss Liu integral that is a type of uncertain integral with respect to Liu process. In addition, the fundamental theorem of uncertain calculus will be proved in this chapter from which the techniques of chain rule, change of variables, and integration by parts are also derived.

Uncertain Differential Equation

Uncertain differential equation is a type of differential equation involving uncertain processes. Chapter 14 will discuss the existence, uniqueness and stability of solutions of uncertain differential equations, and will introduce Yao-Chen formula that represents the solution of an uncertain differential equation by a family of solutions of ordinary differential equations. On the basis of this formula, some formulas to calculate extreme value, first hitting time, and time integral of solution are provided. Furthermore, some numerical methods for solving general uncertain differential equations are designed.

Uncertain Finance

As applications of uncertain differential equation, Chapter 15 will discuss uncertain stock model, uncertain interest rate model, and uncertain currency model.

Uncertain Statistics

Uncertain statistics is a methodology for collecting and interpreting expert's experimental data by uncertainty theory. Chapter 16 will present a questionnaire survey for collecting expert's experimental data. In order to determine uncertainty distributions and membership functions from those expert's experimental data, Chapter 16 will also introduce linear interpolation method, principle of least squares, method of moments, and Delphi method. In addition, uncertain regression analysis and uncertain time series analysis are also introduced when the imprecise observations are characterized in terms of uncertain variables.

Law of Truth Conservation

The law of excluded middle tells us that a proposition is either true or false, and the law of contradiction tells us that a proposition cannot be both true and false. In the state of indeterminacy, some people said, the law of excluded middle and the law of contradiction are no longer valid because the truth degree of a proposition is no longer 0 or 1. I cannot gainsay this viewpoint to a certain extent. But it does not mean that you might "go as you please". *The truth values of a proposition and its negation should sum to unity.* This is the law of truth conservation that is weaker than the law of excluded middle and the law of contradiction. Furthermore, the law of truth conservation agrees with the law of excluded middle and the law of contradiction when the uncertainty vanishes.

Maximum Uncertainty Principle

An event has no uncertainty if its uncertain measure is 1 because we may believe that the event happens. An event has no uncertainty too if its uncertain measure is 0 because we may believe that the event does not happen. An event is the most uncertain if its uncertain measure is 0.5 because the event and its complement may be regarded as "equally likely". In practice, if there is no information about the uncertain measure of an event, we should assign 0.5 to it. Sometimes, only partial information is available. In this case, the value of uncertain measure may be specified in some range. What value does the uncertain measure take? *For any event, if there are multiple reasonable values that an uncertain measure may take, then the value as close to 0.5 as possible is assigned to the event.* This is the maximum uncertainty principle.

Matlab Uncertainty Toolbox

Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) is a collection of functions built on Matlab for many methods of uncertainty theory, including uncertain programming, uncertain risk analysis, uncertain reliability analysis, uncertain logic, uncertain inference, uncertain differential equation, uncertain statistics, scheduling, logistics, data mining, control, and finance.

Lecture Slides

If you need lecture slides for uncertainty theory, please download them from the website at http://orsc.edu.cn/liu/resources.htm.

Uncertainty Theory Online

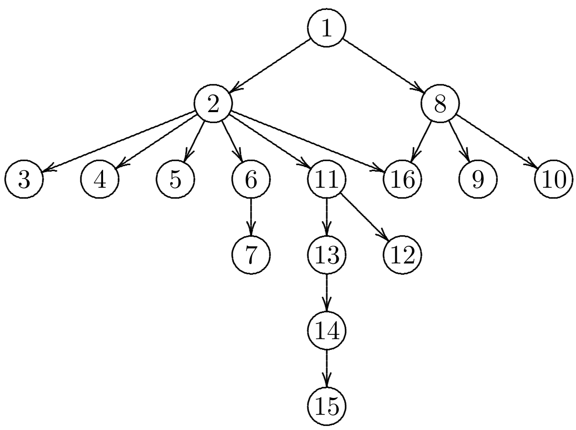
If you want to read more books, dissertations and papers related to uncertainty theory, please visit the website at http://orsc.edu.cn/online.

Purpose

The purpose is to equip the readers with a branch of mathematics to deal with belief degrees. The textbook is suitable for researchers, engineers, and students in the field of mathematics, information science, operations research, industrial engineering, computer science, artificial intelligence, automation, economics, and management science.

A Guide for the Readers

The readers are not required to read the book from cover to cover. The logic dependence of chapters is illustrated by the figure below.



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Chapter 0

Introduction

Real decisions are usually made in the state of indeterminacy. To rationally deal with indeterminacy, there exist two mathematical systems, one is probability theory (Kolmogorov, 1933) and the other is uncertainty theory (Liu, 2007). Probability theory is a branch of mathematics for modelling frequencies, while uncertainty theory is a branch of mathematics for modelling belief degrees.

What is indeterminacy? What is frequency? What is belief degree? This chapter will answer these questions, and show in what situation we should use probability theory and in what situation we should use uncertainty theory. Finally, it is concluded that a rational man behaves as if he used uncertainty theory.

0.1 Indeterminacy

By *indeterminacy* we mean the phenomena whose outcomes cannot be exactly predicted in advance. For example, we cannot exactly predict which face will appear before we toss dice. Thus "tossing dice" is a type of indeterminate phenomenon. As another example, we cannot exactly predict tomorrow's stock price. That is, "stock price" is also a type of indeterminate phenomenon. Some other instances of indeterminacy include "roulette wheel", "product lifetime", "market demand", "bridge strength", "travel distance", etc.

Indeterminacy is absolute, while determinacy is relative. This is the reason why we say real decisions are usually made in the state of indeterminacy. How to model indeterminacy is thus an important research subject in not only mathematics but also science and engineering.

In order to describe an indeterminate quantity (e.g. stock price), what we need is a "distribution function" representing the degree that the quantity falls into the left side of the current point. Such a function will always have bigger values as the current point moves from the left to right. See Figure 1.

If the distribution function takes value 0, then it is completely impossible that the quantity falls into the left side of the current point; if the distribution function takes value 1, then it is completely impossible that the quantity falls into the right side; if the distribution function takes value 0.6, then we are 60% sure that the quantity falls into the left side and 40% sure that the quantity falls into the right side.

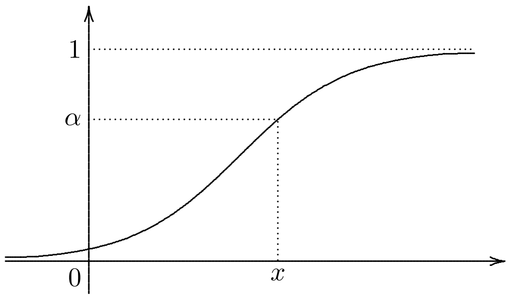


Figure 1: Distribution function

In order to find a distribution function for some indeterminate quantity, personally I think there exist only two ways, one is *frequency* generated by samples (i.e., historical data), and the other is *belief degree* evaluated by domain experts. Could you imagine a third way?

0.2 Frequency

Assume we have collected a set of samples for some indeterminate quantity (e.g. stock price). By *cumulative frequency* we mean a function representing the percentage of all samples that fall into the left side of the current point. It is clear that the cumulative frequency looks like a step function in Figure 2.

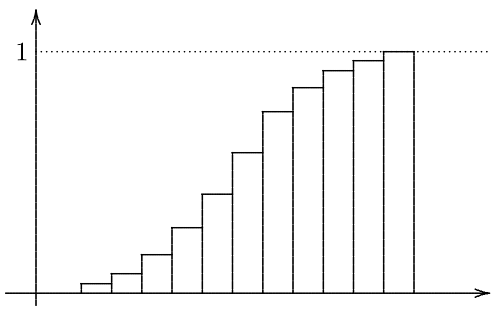


Figure 2: Cumulative frequency histogram

Frequency is a factual property of indeterminate quantity, and does not change with our state of knowledge and preference. In other words, the frequency in the long run exists and is relatively invariant, no matter if it is observed by us.

Probability theory is applicable when samples are available

The study of probability theory was started by Pascal and Fermat in the 17th century when they succeeded in deriving the exact probabilities for certain gambling problems. After that, probability theory was studied by many researchers. Particularly, a complete axiomatic foundation of probability theory was successfully given by Kolmogorov [69] in 1933. Since then, probability theory has been developed steadily and widely applied in science and engineering.

Keep in mind that a fundamental premise of applying probability theory is that the estimated probability distribution is close enough to the long-run cumulative frequency. Otherwise, the law of large numbers is no longer valid and probability theory is no longer applicable.

When the sample size is large enough, it is possible for us to believe the estimated probability distribution is close enough to the long-run cumulative frequency. In this case, there is no doubt that probability theory is the only legitimate approach to deal with our problems on the basis of the estimated probability distributions.

However, in many cases, no samples are available to estimate a probability distribution. What can we do in this situation? Perhaps we have no choice but to invite some domain experts to evaluate the belief degree that each event will happen.

0.3 Belief Degree

Belief degrees are familiar to all of us. The object of belief is an event (i.e., a proposition). For example, "the sun will rise tomorrow", "it will be sunny next week", and "John is a young man" are all instances of object of belief. A *belief degree* represents the strength with which we believe the event will happen. If we completely believe the event will happen, then the belief degree is 1 (complete belief). If we think it is completely impossible, then the belief degree is 0 (complete disbelief). If the event and its complementary event are equally likely, then the belief degree for the event is 0.5, and that for the complementary event is also 0.5. Generally, we will assign a number between 0 and 1 to the belief degree for each event. The higher the belief degree is, the more strongly we believe the event will happen.

Assume a box contains 100 balls, each of which is known to be either red or black, but we do not know how many of the balls are red and how many are black. In this case, it is impossible for us to determine the probability of drawing a red ball. However, the belief degree can be evaluated by us. For example, the belief degree for drawing a red ball is 0.5 because "drawing a red ball" and "drawing a black ball" are equally likely. Besides, the belief degree for drawing a black ball is also 0.5.

The belief degree depends heavily on the personal knowledge (even including preference) concerning the event. When the personal knowledge changes, the belief degree changes too.

Belief Degree Function

How do we describe an indeterminate quantity (e.g. bridge strength)? It is clear that a single belief degree is absolutely not enough. Do we need to know the belief degrees for all possible events? The answer is negative. In fact, what we need is a *belief degree function* that represents the degree with which we believe the indeterminate quantity falls into the left side of the current point.

For example, if we believe the indeterminate quantity completely falls into the left side of the current point, then the belief degree function takes value 1; if we think it completely falls into the right side, then the belief degree function takes value 0. Generally, a belief degree function takes values between 0 and 1, and has bigger values as the current point moves from the left to right. See Figure 3.

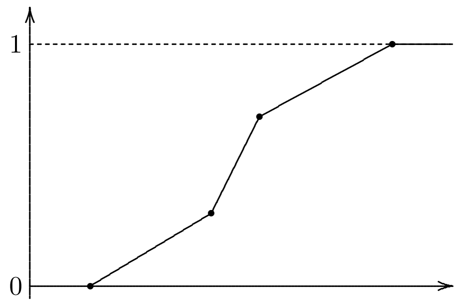


Figure 3: Belief degree function

How to obtain belief degrees

Consider a bridge and its strength. At first, we have to admit that no destructive experiment is allowed for the bridge. Thus we have no samples about the bridge strength. In this case, there do not exist any statistical methods to estimate its probability distribution. How do we deal with it? It seems that we have no choice but to invite some bridge engineers to evaluate the belief degrees about the bridge strength. In practice, it is almost impossible for the bridge engineers to give a perfect description of the belief degrees of all possible events. Instead, they can only provide some subjective judgments about the bridge strength. As a simple example, we assume a consultation process is as follows:

(Q) What do you think is the bridge strength?
(A) I think the bridge strength is between 80 and 120 tons.

What belief degrees can we derive from the answer of the bridge engineer? First, we may have an inference:

(i) I am 100% sure that the bridge strength is less than 120 tons.

This means the belief degree of "the bridge strength being less than 120 tons" is 1. Thus we have an expert's experimental data $(120, 1)$. Furthermore, we may have another inference:

(ii) I am 100% sure that the bridge strength is greater than 80 tons.

This statement gives a belief degree that the bridge strength falls into the right side of 80 tons. We need translate it to a statement about the belief degree that the bridge strength falls into the left side of 80 tons:

(ii′) I am 0% sure that the bridge strength is less than 80 tons.

Although the statement (ii′) sounds strange to us, it is indeed equivalent to the statement (ii). Thus we have another expert's experimental data $(80, 0)$.

Until now we have acquired two expert's experimental data $(80, 0)$ and $(120, 1)$ about the bridge strength. Could we infer the belief degree $\Phi(x)$ that the bridge strength falls into the left side of the point x? The answer is affirmative. For example, a reasonable value is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 80 \\ (x-80)/40, & \text{if } 80 \le x \le 120 \\ 1, & \text{if } x > 120. \end{cases} \tag{1}$$

See Figure 4. From the function $\Phi(x)$, we may infer that the belief degree of "the bridge strength being less than 90 tons" is 0.25. In other words, it is reasonable to infer that "I am 25% sure that the bridge strength is less than 90 tons", or equivalently "I am 75% sure that the bridge strength is greater than 90 tons".

All belief degrees are wrong, but some are useful

Different people may hold different belief degrees. Perhaps some readers may ask which belief degree is correct. Liu [94] answered that *all belief degrees are wrong, but some are useful.* A belief degree becomes "correct" only when it is close enough to the frequency of the indeterminate quantity. However, usually we cannot make it to that.

Through a lot of surveys, Kahneman and Tversky [64] showed that *human beings usually overweight unlikely events.* From another side, Liu [94] showed

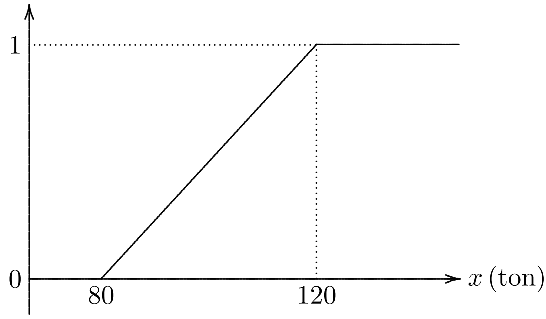


Figure 4: Belief degree function of “the bridge strength”

that *human beings usually estimate a much wider range of values than the object actually takes.* This conservatism of human beings makes the belief degrees deviate far from the frequency. Thus all belief degrees are wrong compared with its frequency. However, it cannot be denied that those belief degrees are indeed helpful for decision making.

Belief degrees cannot be treated as subjective probability

Can we deal with belief degrees by probability theory? Some people do think so and call it subjective probability. However, Liu [85] declared that it is inappropriate to model belief degrees by probability theory because it may lead to counterintuitive results.

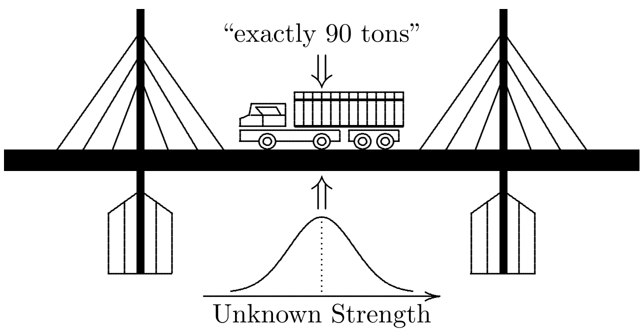


Figure 5: A Truck is Crossing over a Bridge

Consider a counterexample presented by Liu [85]. Assume there is one truck and 50 bridges in an experiment. Also assume the weight of the truck is 90 tons and the 50 bridge strengths are iid uniform random variables on $[95, 110]$ in tons. For simplicity, suppose a bridge collapses whenever its real strength is less than the weight of the truck. Now let us have the truck cross over the 50 bridges one by one. It is easy to verify that

$$\Pr\{\text{“the truck can cross over the 50 bridges”}\} = 1. \tag{2}$$

That is to say, we are 100% sure that the truck can cross over the 50 bridges successfully.

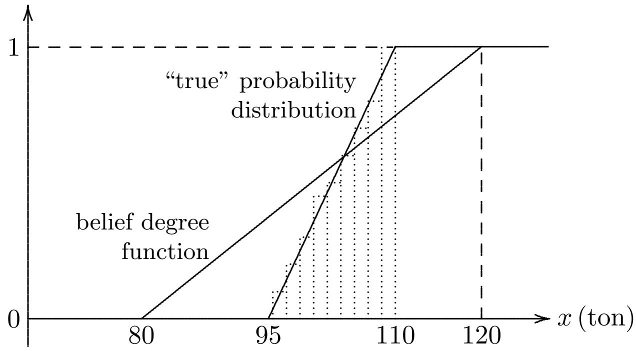


Figure 6: Belief degree function, “true” probability distribution and cumulative frequency histogram of “the bridge strength”

However, when there do not exist any observed samples for the bridge strength at the moment, we have to invite some bridge engineers to evaluate the belief degrees about it. As we stated before, human beings usually estimate a much wider range of values than the bridge strength actually takes because of the conservatism. Assume the belief degree function is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 80 \\ (x-80)/40, & \text{if } 80 \le x \le 120 \\ 1, & \text{if } x > 120. \end{cases} \tag{3}$$

See Figure 6. Let us imagine what will happen if the belief degree function is treated as a probability distribution. At first, we have to regard the 50 bridge strengths as iid uniform random variables on $[80, 120]$ in tons. If we have the truck cross over the 50 bridges one by one, then we immediately have

$$\Pr\{\text{“the truck can cross over the 50 bridges”}\} = 0.75^{50} \approx 0. \tag{4}$$

Thus it is almost impossible that the truck crosses over the 50 bridges successfully. Unfortunately, the results (2) and (4) are at opposite poles. This example shows that, by inappropriately using probability theory, a sure event becomes an impossible one. The error seems intolerable for us. Hence the belief degrees cannot be treated as subjective probability.

A possible proposition cannot be judged impossible

During information processing, we should follow such a basic principle that a possible proposition cannot be judged impossible (Liu [85]). In other words, if a proposition is possibly true, then its truth value should not be zero. Likewise, if a proposition is possibly false, then its truth value should not be unity.

In the example of truck-cross-over-bridge, a completely true proposition is judged completely false by probability theory. This means using probability theory violates the above-mentioned principle, and therefore probability theory is not appropriate to model belief degrees. In other words, belief degrees do not follow the laws of probability theory.

Uncertainty theory is able to model belief degrees

In order to rationally deal with personal belief degrees, uncertainty theory was founded by Liu [76] in 2007 and subsequently studied by many researchers. Nowadays, uncertainty theory has become a branch of mathematics for modelling belief degrees.

Liu [85] declared that uncertainty theory is the only legitimate approach when only belief degrees are available. If we believe the estimated uncertainty distribution is close enough to the belief degrees hidden in the mind of the domain experts, then we may use uncertainty theory to deal with our own problems on the basis of the estimated uncertainty distributions.

Let us reconsider the example of truck-cross-over-bridge by uncertainty theory. If the belief degree function is regarded as a linear uncertainty distribution on $[80, 120]$ in tons, then we immediately have

$$\mathcal{M}\{\text{"the truck can cross over the 50 bridges"}\} = 0.75. \tag{5}$$

That is to say, we are 75% sure that the truck can cross over the 50 bridges successfully. Here the degree 75% does not achieve up to the true value 100%. But the error is caused by the difference between belief degree and frequency, and is not further magnified by uncertainty theory.

0.4 Summary

In order to model indeterminacy, many theories have been invented. What theories are considered acceptable? Personally I think *an acceptable theory should be not only theoretically self-consistent but also the best among others for solving at least one practical problem.* On the basis of this principle, I may conclude that there exist two mathematical systems, one is probability theory and the other is uncertainty theory. It is emphasized that probability theory is only applicable to modelling frequencies, and uncertainty theory is only applicable to modelling belief degrees. In other words, frequency is the empirical basis of probability theory, while belief degree is the empirical basis of uncertainty theory. Keep in mind that using uncertainty theory to model frequency may produce a crude result, while using probability theory to model belief degree may produce a big disaster.

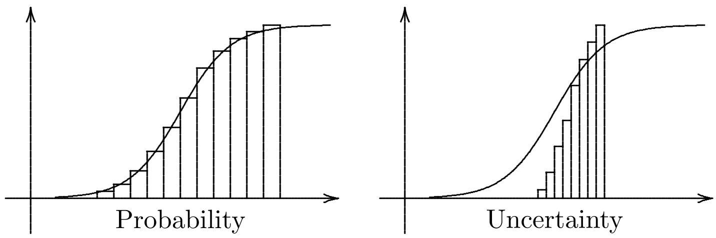


Figure 7: When the sample size is large enough, the estimated probability distribution (left curve) may be close enough to the cumulative frequency (left histogram). In this case, probability theory is the only legitimate approach. When the belief degrees are available (no samples), the estimated uncertainty distribution (right curve) usually deviates far from the cumulative frequency (right histogram but unknown). In this case, uncertainty theory is the only legitimate approach.

However, single-variable system is an exception. When there exists one and only one indeterminate variable in a real system, probability theory and uncertainty theory will produce the same result because product measure is not used. In this case, frequency may be modeled by uncertainty theory while belief degree may be modeled by probability theory. Both are indifferent.

Since belief degrees are usually wrong compared with frequency, the gap between belief degree and frequency always exists. Such an error is likely to be further magnified if the belief degree is regarded as subjective probability. Fortunately, uncertainty theory can successfully avoid turning small errors to large ones.

Savage [132] said *a rational man behaves as if he used subjective probabilities.* However, usually, we cannot make it to that. Liu [94] said *a rational man behaves as if he used uncertainty theory.* In other words, a rational man is expected to hold belief degrees that follow the laws of uncertainty theory rather than probability theory.

Chapter 1

Uncertain Measure

Uncertainty theory was founded by Liu [76] in 2007 and subsequently studied by many researchers. Nowadays uncertainty theory has become a branch of mathematics for modelling belief degrees. This chapter will provide normality, duality, subadditivity and product axioms of uncertainty theory. From those four axioms, this chapter will also introduce an uncertain measure that is a fundamental concept in uncertainty theory. In addition, product uncertain measure and conditional uncertain measure will be explored at the end of this chapter.

1.1 Measurable Space

From the mathematical viewpoint, uncertainty theory is essentially an alternative theory of measure. Thus uncertainty theory should begin with a measurable space. In order to learn it, let us introduce algebra, σ-algebra, measurable set, Borel algebra, Borel set, and measurable function. The main results in this section are well-known. For this reason the credit references are not provided. You may skip this section if you are familiar with them.

Definition 1.1 *Let Γ be a nonempty set (sometimes called universal set). A collection $\mathcal{L}$ consisting of subsets of Γ is called an algebra over Γ if the following three conditions hold: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, then*

$$\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}. \tag{1.1}$$

The collection $\mathcal{L}$ is called a σ-algebra over Γ if the condition (c) is replaced with closure under countable union, i.e., when $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, we have

$$\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}. \tag{1.2}$$

Example 1.1: The collection $\{\emptyset, \Gamma\}$ is the smallest σ-algebra over Γ, and the power set (i.e., all subsets of Γ) is the largest σ-algebra.

Example 1.2: Let Λ be a proper nonempty subset of Γ. Then $\{\emptyset, \Lambda, \Lambda^c, \Gamma\}$ is a σ-algebra over Γ.

Example 1.3: Let $\mathcal{L}$ be the collection of all finite disjoint unions of all intervals of the form

$$(-\infty, a], \quad (a, b], \quad (b, \infty), \quad \emptyset. \tag{1.3}$$

Then $\mathcal{L}$ is an algebra over $\Re$ (the set of real numbers), but not a σ-algebra because $\Lambda_i = (0, (i-1)/i] \in \mathcal{L}$ for all i but

$$\bigcup_{i=1}^{\infty} \Lambda_i = (0, 1) \notin \mathcal{L}. \tag{1.4}$$

Example 1.4: A σ-algebra $\mathcal{L}$ is closed under countable union, countable intersection, difference, and limit. That is, if $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, then

$$\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}; \quad \bigcap_{i=1}^{\infty} \Lambda_i \in \mathcal{L}; \quad \Lambda_1 \setminus \Lambda_2 \in \mathcal{L}; \quad \lim_{i \to \infty} \Lambda_i \in \mathcal{L}. \tag{1.5}$$

Definition 1.2 *Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. Then $(\Gamma, \mathcal{L})$ is called a measurable space, and any element in $\mathcal{L}$ is called a measurable set.*

Example 1.5: Let $\Re$ be the set of real numbers. Then $\mathcal{L} = \{\emptyset, \Re\}$ is a σ-algebra over $\Re$. Thus $(\Re, \mathcal{L})$ is a measurable space. Note that there exist only two measurable sets in this space, one is $\emptyset$ and another is $\Re$. Keep in mind that the intervals like $[0, 1]$ and $(0, +\infty)$ are not measurable in this space!

Example 1.6: Let $\Gamma = \{a, b, c\}$. Then $\mathcal{L} = \{\emptyset, \{a\}, \{b, c\}, \Gamma\}$ is a σ-algebra over Γ. Thus $(\Gamma, \mathcal{L})$ is a measurable space. Furthermore, $\{a\}$ and $\{b, c\}$ are measurable sets in this space, but $\{b\}, \{c\}, \{a, b\}, \{a, c\}$ are not.

Definition 1.3 *The smallest σ-algebra $\mathcal{B}$ containing all open intervals is called the Borel algebra over the set of real numbers, and any element in $\mathcal{B}$ is called a Borel set.*

Example 1.7: It has been proved that intervals, open sets, closed sets, rational numbers, and irrational numbers are all Borel sets.

Example 1.8: There exists a non-Borel set over $\Re$. Let $[a]$ represent the set of all rational numbers plus a. Note that if $a_1 - a_2$ is not a rational number, then $[a_1]$ and $[a_2]$ are disjoint sets. Thus $\Re$ is divided into an infinite number of those disjoint sets. Let A be a new set containing precisely one element from them. Then A is not a Borel set.

Definition 1.4 *A function ξ from a measurable space $(\Gamma, \mathcal{L})$ to the set of real numbers is said to be measurable if*

$$\xi^{-1}(B) = \{\gamma \in \Gamma \,|\, \xi(\gamma) \in B\} \in \mathcal{L} \tag{1.6}$$

for any Borel set B of real numbers.

Continuous function and monotone function are instances of measurable function. Let $\xi_1, \xi_2, \cdots$ be a sequence of measurable functions. Then the following functions are also measurable:

$$\sup_{1\le i<\infty} \xi_i(\gamma); \quad \inf_{1\le i<\infty} \xi_i(\gamma); \quad \limsup_{i\to\infty} \xi_i(\gamma); \quad \liminf_{i\to\infty} \xi_i(\gamma). \tag{1.7}$$

Especially, if $\lim_{i\to\infty} \xi_i(\gamma)$ exists for each γ, then the limit is also a measurable function.

1.2 Uncertain Measure

Let $(\Gamma, \mathcal{L})$ be a measurable space. Recall that each element Λ in $\mathcal{L}$ is called a measurable set. The first action we take is to rename measurable set as *event* in uncertainty theory. The second action is to define an uncertain measure $\mathcal{M}$ on the σ-algebra $\mathcal{L}$. That is, a number $\mathcal{M}\{\Lambda\}$ will be assigned to each event Λ to indicate the belief degree with which we believe Λ will happen. There is no doubt that the assignment is not arbitrary, and the uncertain measure $\mathcal{M}$ must have certain mathematical properties. In order to rationally deal with belief degrees, Liu [76] suggested the following three axioms:

Axiom 1. *(Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.*

Axiom 2. *(Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.*

Axiom 3. *(Subadditivity Axiom) For every countable sequence of events Λ_1, $\Lambda_2, \cdots$, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{1.8}$$

Remark 1.1: Uncertain measure is interpreted as the personal belief degree (not frequency) of an uncertain event that may happen. It depends on the personal knowledge concerning the event. The uncertain measure will change if the state of knowledge changes.

Remark 1.2: Since "1" means "complete belief" and we cannot be in more belief than "complete belief", the belief degree of any event cannot exceed 1.

Furthermore, the belief degree of the universal set takes value 1 because it is completely believable. Thus the belief degree meets the normality axiom.

Remark 1.3: Duality axiom is in fact an application of the law of truth conservation in uncertainty theory. The property ensures that the uncertainty theory is consistent with the law of excluded middle and the law of contradiction. In addition, the human thinking is always dominated by the duality. For example, if someone tells us that a proposition is true with belief degree 0.6, then all of us will think that the proposition is false with belief degree 0.4.

Remark 1.4: Given two events with known belief degrees, it is frequently asked that how the belief degree for their union is generated from the individuals. Personally, I do not think there exists any rule to make it. A lot of surveys showed that, generally speaking, the belief degree of a union of events is neither the sum of belief degrees of the individual events (e.g. probability measure) nor the maximum (e.g. possibility measure). It seems that there is no explicit relation between the union and individuals except for the subadditivity axiom.

Remark 1.5: Pathology occurs if subadditivity axiom is not assumed. For example, suppose that a universal set contains 3 elements. We define a set function that takes value 0 for each singleton, and 1 for each event with at least 2 elements. Then such a set function satisfies all axioms but subadditivity. Do you think it is strange if such a set function serves as a measure?

Remark 1.6: Although probability measure satisfies the above three axioms, probability theory is not a special case of uncertainty theory because the product probability measure does not satisfy the fourth axiom, namely the product axiom on Page 20.

Definition 1.5 *(Liu [76]) The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.*

Exercise 1.1: Let Γ be a nonempty set. For each subset Λ of Γ, we define

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise.} \end{cases} \tag{1.9}$$

Show that $\mathcal{M}$ is an uncertain measure. (Hint: Verify $\mathcal{M}$ meets the three axioms.)

Exercise 1.2: Let $\Gamma = \{\gamma_1, \gamma_2\}$. It is clear that there exist 4 events in the power set,

$$\mathcal{L} = \{\emptyset,\ \{\gamma_1\},\ \{\gamma_2\},\ \Gamma\}. \tag{1.10}$$

Assume c is a real number with $0 < c < 1$, and define

$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\gamma_1\} = c, \quad \mathcal{M}\{\gamma_2\} = 1 - c, \quad \mathcal{M}\{\Gamma\} = 1.$$

Show that $\mathcal{M}$ is an uncertain measure.

Exercise 1.3: Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. It is clear that there exist 8 events in the power set,

$$\mathcal{L} = \{\emptyset, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}, \{\gamma_1, \gamma_2\}, \{\gamma_1, \gamma_3\}, \{\gamma_2, \gamma_3\}, \Gamma\}. \tag{1.11}$$

Assume c_1, c_2, c_3 are nonnegative numbers satisfying the consistency condition

$$c_i + c_j \le 1 \le c_1 + c_2 + c_3, \quad \forall i \ne j. \tag{1.12}$$

Define

$$\mathcal{M}\{\gamma_1\} = c_1, \quad \mathcal{M}\{\gamma_2\} = c_2, \quad \mathcal{M}\{\gamma_3\} = c_3,$$

$$\mathcal{M}\{\gamma_1, \gamma_2\} = 1 - c_3, \quad \mathcal{M}\{\gamma_1, \gamma_3\} = 1 - c_2, \quad \mathcal{M}\{\gamma_2, \gamma_3\} = 1 - c_1,$$

$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Gamma\} = 1.$$

Show that $\mathcal{M}$ is an uncertain measure.

Exercise 1.4: Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, and let c be a real number with $0.5 \le c < 1$. It is clear that there exist 16 events in the power set. For each subset Λ, define

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ c, & \text{if } \gamma_1 \in \Lambda \ne \Gamma \\ 1 - c, & \text{if } \gamma_1 \notin \Lambda \ne \emptyset. \end{cases} \tag{1.13}$$

Show that $\mathcal{M}$ is an uncertain measure.

Exercise 1.5: Let $\Gamma = \{\gamma_1, \gamma_2, \cdots\}$, and let $c_1, c_2, \cdots$ be nonnegative numbers such that $c_1 + c_2 + \cdots = 1$. For each subset Λ, define

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_i \in \Lambda} c_i. \tag{1.14}$$

Show that $\mathcal{M}$ is an uncertain measure.

Exercise 1.6: Lebesgue measure, named after French mathematician Henri Lebesgue, is the standard way of assigning a length, area or volume to subsets of Euclidean space. For example, the Lebesgue measure of the interval $[a, b]$ of real numbers is the length $b - a$. Let $\Gamma = [0, 1]$, and let $\mathcal{M}$ be the Lebesgue measure. Show that $\mathcal{M}$ is an uncertain measure.

Exercise 1.7: Let Γ be the set of real numbers, and let c be a real number with $0 < c \le 0.5$. For each subset Λ, define

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ c, & \text{if } \Lambda \text{ is upper bounded and } \Lambda \neq \emptyset \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ 1-c, & \text{if } \Lambda^c \text{ is upper bounded and } \Lambda \neq \Gamma \\ 1, & \text{if } \Lambda = \Gamma. \end{cases} \tag{1.15}$$

Show that $\mathcal{M}$ is an uncertain measure.

Exercise 1.8: Suppose that $\lambda(x)$ is a nonnegative function on $\Re$ (the set of real numbers) such that

$$\sup_{x\in\Re} \lambda(x) = 0.5. \tag{1.16}$$

Define a set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{x\in\Lambda} \lambda(x), & \text{if } \sup\limits_{x\in\Lambda} \lambda(x) < 0.5 \\ 1 - \sup\limits_{x\in\Lambda^c} \lambda(x), & \text{if } \sup\limits_{x\in\Lambda} \lambda(x) = 0.5 \end{cases} \tag{1.17}$$

for each subset Λ. Show that $\mathcal{M}$ is an uncertain measure.

Exercise 1.9: Suppose $\rho(x)$ is a nonnegative and integrable function on $\Re$ (the set of real numbers) such that

$$\int_{\Re} \rho(x)\mathrm{d}x \ge 1. \tag{1.18}$$

Define a set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x < 0.5 \\ 1 - \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x < 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{1.19}$$

for each Borel set Λ. Show that $\mathcal{M}$ is an uncertain measure.

Theorem 1.1 *(Monotonicity Theorem) The uncertain measure is a monotone increasing set function. That is, for any events Λ_1 and Λ_2 with $\Lambda_1 \subset \Lambda_2$, we have*

$$\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}. \tag{1.20}$$

Proof: The normality axiom says $\mathcal{M}\{\Gamma\} = 1$, and the duality axiom says $\mathcal{M}\{\Lambda_1^c\} = 1 - \mathcal{M}\{\Lambda_1\}$. Since $\Lambda_1 \subset \Lambda_2$, we have $\Gamma = \Lambda_1^c \cup \Lambda_2$. By using the subadditivity axiom, we obtain

$$1 = \mathcal{M}\{\Gamma\} \le \mathcal{M}\{\Lambda_1^c\} + \mathcal{M}\{\Lambda_2\} = 1 - \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\}.$$

Thus $\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}$.

Theorem 1.2 *The empty set $\emptyset$ always has an uncertain measure zero. That is,*

$$\mathcal{M}\{\emptyset\} = 0. \tag{1.21}$$

Proof: Since $\emptyset = \Gamma^c$ and $\mathcal{M}\{\Gamma\} = 1$, it follows from the duality axiom that

$$\mathcal{M}\{\emptyset\} = 1 - \mathcal{M}\{\Gamma\} = 1 - 1 = 0.$$

Theorem 1.3 *The uncertain measure takes values between 0 and 1. That is, for any event Λ, we have*

$$0 \le \mathcal{M}\{\Lambda\} \le 1. \tag{1.22}$$

Proof: It follows from the monotonicity theorem that $0 \le \mathcal{M}\{\Lambda\} \le 1$ because $\emptyset \subset \Lambda \subset \Gamma$ and $\mathcal{M}\{\emptyset\} = 0$, $\mathcal{M}\{\Gamma\} = 1$.

Theorem 1.4 *Let $\Lambda_1, \Lambda_2, \cdots$ be a sequence of events with $\mathcal{M}\{\Lambda_i\} \to 0$ as $i \to \infty$. Then for any event Λ, we have*

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda \cup \Lambda_i\} = \lim_{i\to\infty} \mathcal{M}\{\Lambda\backslash\Lambda_i\} = \mathcal{M}\{\Lambda\}. \tag{1.23}$$

Especially, an uncertain measure remains unchanged if the event is enlarged or reduced by an event with uncertain measure zero.

Proof: It follows from the monotonicity theorem and subadditivity axiom that

$$\mathcal{M}\{\Lambda\} \le \mathcal{M}\{\Lambda \cup \Lambda_i\} \le \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda_i\}$$

for each i. Thus we get $\mathcal{M}\{\Lambda \cup \Lambda_i\} \to \mathcal{M}\{\Lambda\}$ by using $\mathcal{M}\{\Lambda_i\} \to 0$. Since $(\Lambda\backslash\Lambda_i) \subset \Lambda \subset ((\Lambda\backslash\Lambda_i) \cup \Lambda_i)$, we have

$$\mathcal{M}\{\Lambda\backslash\Lambda_i\} \le \mathcal{M}\{\Lambda\} \le \mathcal{M}\{\Lambda\backslash\Lambda_i\} + \mathcal{M}\{\Lambda_i\}.$$

Hence $\mathcal{M}\{\Lambda\backslash\Lambda_i\} \to \mathcal{M}\{\Lambda\}$ by using $\mathcal{M}\{\Lambda_i\} \to 0$.

Theorem 1.5 *(Asymptotic Theorem) For any events $\Lambda_1, \Lambda_2, \cdots$, we have*

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} > 0, \quad \textit{if } \Lambda_i \uparrow \Gamma, \tag{1.24}$$

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} < 1, \quad \textit{if } \Lambda_i \downarrow \emptyset. \tag{1.25}$$

Proof: Assume $\Lambda_i \uparrow \Gamma$. Since $\Gamma = \cup_i \Lambda_i$, it follows from the subadditivity axiom that

$$1 = \mathcal{M}\{\Gamma\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Since $\mathcal{M}\{\Lambda_i\}$ is increasing with respect to i, we have $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} > 0$. If $\Lambda_i \downarrow \emptyset$, then $\Lambda_i^c \uparrow \Gamma$. It follows from the first inequality and the duality axiom that

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = 1 - \lim_{i\to\infty} \mathcal{M}\{\Lambda_i^c\} < 1.$$

The theorem is proved.

Example 1.9: Assume Γ is the set of real numbers. Let α be a number with $0 < \alpha \le 0.5$. Define an uncertain measure as follows,

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ \alpha, & \text{if } \Lambda \text{ is upper bounded and } \Lambda \ne \emptyset \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ 1-\alpha, & \text{if } \Lambda^c \text{ is upper bounded and } \Lambda \ne \Gamma \\ 1, & \text{if } \Lambda = \Gamma. \end{cases} \tag{1.26}$$

(i) Write $\Lambda_i = (-\infty, i]$ for $i = 1, 2, \cdots$ Then $\Lambda_i \uparrow \Gamma$ and $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = \alpha$.
(ii) Write $\Lambda_i = [i, +\infty)$ for $i = 1, 2, \cdots$ Then $\Lambda_i \downarrow \emptyset$ and $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = 1 - \alpha$.

1.3 Uncertainty Space

Definition 1.6 *(Liu [76]) Let Γ be a nonempty set, let $\mathcal{L}$ be a σ-algebra over Γ, and let $\mathcal{M}$ be an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.*

Example 1.10: Let Γ be a two-point set $\{\gamma_1, \gamma_2\}$, let $\mathcal{L}$ be the power set of $\{\gamma_1, \gamma_2\}$, and let $\mathcal{M}$ be an uncertain measure determined by $\mathcal{M}\{\gamma_1\} = 0.6$ and $\mathcal{M}\{\gamma_2\} = 0.4$. Then $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space.

Example 1.11: Let Γ be a three-point set $\{\gamma_1, \gamma_2, \gamma_3\}$, let $\mathcal{L}$ be the power set of $\{\gamma_1, \gamma_2, \gamma_3\}$, and let $\mathcal{M}$ be an uncertain measure determined by $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$ and $\mathcal{M}\{\gamma_3\} = 0.2$. Then $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space.

Example 1.12: Let Γ be the interval $[0, 1]$, let $\mathcal{L}$ be the Borel algebra over $[0, 1]$, and let $\mathcal{M}$ be the Lebesgue measure. Then $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space.

For practical purposes, the study of uncertainty spaces is sometimes restricted to complete uncertainty spaces.

Definition 1.7 *(Liu [94]) An uncertainty space* $(\Gamma,\mathcal{L},\mathcal{M})$ *is called complete if for any* $\Lambda_1,\Lambda_2\in\mathcal{L}$ *with* $\mathcal{M}\{\Lambda_1\}=\mathcal{M}\{\Lambda_2\}$ *and any subset* A *with* $\Lambda_1\subset A\subset\Lambda_2$*, one has* $A\in\mathcal{L}$*. In this case, we also have*

$$\mathcal{M}\{A\}=\mathcal{M}\{\Lambda_1\}=\mathcal{M}\{\Lambda_2\}. \tag{1.27}$$

Exercise 1.10: Let $(\Gamma,\mathcal{L},\mathcal{M})$ be a complete uncertainty space, and let Λ be an event with $\mathcal{M}\{\Lambda\}=0$. Show that A is an event and $\mathcal{M}\{A\}=0$ whenever $A\subset\Lambda$.

Exercise 1.11: Let $(\Gamma,\mathcal{L},\mathcal{M})$ be a complete uncertainty space, and let Λ be an event with $\mathcal{M}\{\Lambda\}=1$. Show that A is an event and $\mathcal{M}\{A\}=1$ whenever $A\supset\Lambda$.

Definition 1.8 *(Gao [40]) An uncertainty space* $(\Gamma,\mathcal{L},\mathcal{M})$ *is called continuous if for any events* $\Lambda_1,\Lambda_2,\cdots$*, we have*

$$\mathcal{M}\left\{\lim_{i\to\infty}\Lambda_i\right\}=\lim_{i\to\infty}\mathcal{M}\{\Lambda_i\} \tag{1.28}$$

provided that $\lim_{i\to\infty}\Lambda_i$ exists.

Exercise 1.12: Show that an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ is always continuous if Γ consists of a finite number of points.

Exercise 1.13: Let $\Gamma=[0,1]$, let $\mathcal{L}$ be the Borel algebra over Γ, and let $\mathcal{M}$ be the Lebesgue measure. Show that $(\Gamma,\mathcal{L},\mathcal{M})$ is a continuous uncertainty space.

Exercise 1.14: Let $\Gamma=[0,1]$, and let $\mathcal{L}$ be the power set over Γ. For each subset Λ of Γ, define

$$\mathcal{M}\{\Lambda\}=\begin{cases}0, & \text{if } \Lambda=\emptyset\\ 1, & \text{if } \Lambda=\Gamma\\ 0.5, & \text{otherwise.}\end{cases} \tag{1.29}$$

Show that $(\Gamma,\mathcal{L},\mathcal{M})$ is a discontinuous uncertainty space.

1.4 Product Uncertain Measure

Product uncertain measure was defined by Liu [79] in 2009, thus producing the fourth axiom of uncertainty theory. Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\cdots$ Write

$$\Gamma=\Gamma_1\times\Gamma_2\times\cdots \tag{1.30}$$

that is the set of all ordered tuples of the form $(\gamma_1, \gamma_2, \cdots)$, where $\gamma_k \in \Gamma_k$ for $k = 1, 2, \cdots$ A measurable rectangle in Γ is a set

$$\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \tag{1.31}$$

where $\Lambda_k \in \mathcal{L}_k$ for $k = 1, 2, \cdots$ The smallest σ-algebra containing all measurable rectangles of Γ is called the product σ-algebra, denoted by

$$\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \tag{1.32}$$

Then the product uncertain measure $\mathcal{M}$ on the product σ-algebra $\mathcal{L}$ is defined by the following product axiom (Liu [79]).

Axiom 4. *(Product Axiom) Let* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ *be uncertainty spaces for* $k = 1, 2, \cdots$ *The product uncertain measure* $\mathcal{M}$ *is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{1.33}$$

where Λ_k *are arbitrarily chosen events from* $\mathcal{L}_k$ *for* $k = 1, 2, \cdots$, *respectively.*

Remark 1.7: Note that (1.33) defines a product uncertain measure only for rectangles. How do we extend the uncertain measure $\mathcal{M}$ from the class of rectangles to the product σ-algebra $\mathcal{L}$? For each event $\Lambda \in \mathcal{L}$, we have

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if } \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if } \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, \qquad \text{otherwise.} \end{cases} \tag{1.34}$$

Remark 1.8: The sum of the uncertain measures of the maximum rectangles in Λ and Λ^c is always less than or equal to 1, i.e.,

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} + \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \le 1.$$

This means that at most one of

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \quad \text{and} \quad \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}$$

is greater than 0.5. Thus the expression (1.34) is reasonable.

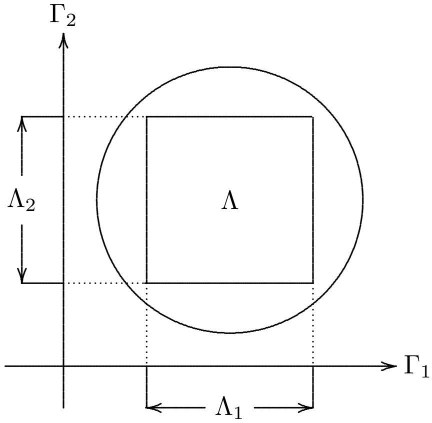


Figure 1.1: Extension from Rectangles to Product σ-Algebra. The uncertain measure of Λ (the disk) is essentially the acreage of its inscribed rectangle $\Lambda_1\times\Lambda_2$ if it is greater than 0.5. Otherwise, we have to examine its complement Λ^c. If the inscribed rectangle of Λ^c is greater than 0.5, then $\mathcal{M}\{\Lambda^c\}$ is just its inscribed rectangle and $\mathcal{M}\{\Lambda\} = 1 - \mathcal{M}\{\Lambda^c\}$. If there does not exist an inscribed rectangle of Λ or Λ^c greater than 0.5, then we set $\mathcal{M}\{\Lambda\} = 0.5$.

Remark 1.9: It is clear that for each $\Lambda \in \mathcal{L}$, the uncertain measure $\mathcal{M}\{\Lambda\}$ defined by (1.34) takes possible values on the interval

$$\left[\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\},\ 1-\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\right].$$

Thus (1.34) coincides with the maximum uncertainty principle (Liu [76]), that is, $\mathcal{M}\{\Lambda\}$ takes the value as close to 0.5 as possible within the above interval.

Remark 1.10: If the sum of the uncertain measures of the maximum rectangles in Λ and Λ^c is just 1, i.e.,

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}+\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}=1,$$

then the product uncertain measure (1.34) is simplified as

$$\mathcal{M}\{\Lambda\}=\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}. \tag{1.35}$$

Exercise 1.15: Let $(\Gamma_1,\mathcal{L}_1,\mathcal{M}_1)$ be the interval $[0,1]$ with Borel algebra and Lebesgue measure, and let $(\Gamma_2,\mathcal{L}_2,\mathcal{M}_2)$ be also the interval $[0,1]$ with Borel algebra and Lebesgue measure. Then

$$\Lambda=\{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2\,|\,\gamma_1+\gamma_2\le 1\} \tag{1.36}$$

is an event on the product uncertainty space $(\Gamma_1,\mathcal{L}_1,\mathcal{M}_1)\times(\Gamma_2,\mathcal{L}_2,\mathcal{M}_2)$. Show that

$$\mathcal{M}\{\Lambda\}=\frac{1}{2}. \tag{1.37}$$

Exercise 1.16: Let $(\Gamma_1,\mathcal{L}_1,\mathcal{M}_1)$ be the interval $[0,1]$ with Borel algebra and Lebesgue measure, and let $(\Gamma_2,\mathcal{L}_2,\mathcal{M}_2)$ be also the interval $[0,1]$ with Borel algebra and Lebesgue measure. Then

$$\Lambda=\left\{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2\,|\,(\gamma_1-0.5)^2+(\gamma_2-0.5)^2\le 0.5^2\right\} \tag{1.38}$$

is an event on the product uncertainty space $(\Gamma_1,\mathcal{L}_1,\mathcal{M}_1)\times(\Gamma_2,\mathcal{L}_2,\mathcal{M}_2)$. (i) Show that

$$\mathcal{M}\{\Lambda\}=\frac{1}{\sqrt{2}}. \tag{1.39}$$

(ii) From the above result we derive $\mathcal{M}\{\Lambda^c\}=1-1/\sqrt{2}$. Please find a rectangle $\Lambda_1\times\Lambda_2$ in Λ^c such that $\mathcal{M}\{\Lambda_1\times\Lambda_2\}=1-1/\sqrt{2}$.

Theorem 1.6 *(Peng-Iwamura [121]) The product uncertain measure defined by (1.34) is an uncertain measure.*

Proof: In order to prove that the product uncertain measure (1.34) is indeed an uncertain measure, we should verify that the product uncertain measure satisfies the normality, duality and subadditivity axioms.

STEP 1: The product uncertain measure is clearly normal, i.e., $\mathcal{M}\{\Gamma\}=1$.

STEP 2: We prove the duality, i.e., $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

Then we immediately have

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}<0.5.$$

It follows from (1.34) that

$$\mathcal{M}\{\Lambda\}=\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\},$$

$$\mathcal{M}\{\Lambda^c\}=1-\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset(\Lambda^c)^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}=1-\mathcal{M}\{\Lambda\}.$$

The duality is proved. Case 2: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

This case may be proved by a similar process. Case 3: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le 0.5$$

and

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le 0.5.$$

It follows from (1.34) that $\mathcal{M}\{\Lambda\}=\mathcal{M}\{\Lambda^c\}=0.5$ which proves the duality.

STEP 3: Let us prove that $\mathcal{M}$ is an increasing set function. Suppose Λ and Δ are two events in $\mathcal{L}$ with $\Lambda\subset\Delta$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

Then

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\ge\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

It follows from (1.34) that $\mathcal{M}\{\Lambda\}\le\mathcal{M}\{\Delta\}$. Case 2: Assume

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}>0.5.$$

Then

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\ge\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}>0.5.$$

Thus

$$\begin{aligned}\mathcal{M}\{\Lambda\}&=1-\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\\&\le 1-\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}=\mathcal{M}\{\Delta\}.\end{aligned}$$

Case 3: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le 0.5$$

and

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\le 0.5.$$

Then

$$\mathcal{M}\{\Lambda\}\le 0.5\le 1-\mathcal{M}\{\Delta^c\}=\mathcal{M}\{\Delta\}.$$

STEP 4: Finally, we prove the subadditivity of $\mathcal{M}$. For simplicity, we only prove the case of two events Λ and Δ. The argument breaks down into three cases. Case 1: Assume $\mathcal{M}\{\Lambda\} < 0.5$ and $\mathcal{M}\{\Delta\} < 0.5$. For any given $\varepsilon > 0$, there are two rectangles

$$\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c, \quad \Delta_1 \times \Delta_2 \times \cdots \subset \Delta^c$$

such that

$$1 - \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \le \mathcal{M}\{\Lambda\} + \varepsilon/2,$$
$$1 - \min_{1 \le k < \infty} \mathcal{M}_k\{\Delta_k\} \le \mathcal{M}\{\Delta\} + \varepsilon/2.$$

Note that

$$(\Lambda_1 \cap \Delta_1) \times (\Lambda_2 \cap \Delta_2) \times \cdots \subset (\Lambda \cup \Delta)^c.$$

It follows from the duality and subadditivity axioms that

$$\begin{aligned}
\mathcal{M}_k\{\Lambda_k \cap \Delta_k\} &= 1 - \mathcal{M}_k\{(\Lambda_k \cap \Delta_k)^c\} = 1 - \mathcal{M}_k\{\Lambda_k^c \cup \Delta_k^c\} \\
&\ge 1 - (\mathcal{M}_k\{\Lambda_k^c\} + \mathcal{M}_k\{\Delta_k^c\}) \\
&= 1 - (1 - \mathcal{M}_k\{\Lambda_k\}) - (1 - \mathcal{M}_k\{\Delta_k\}) \\
&= \mathcal{M}_k\{\Lambda_k\} + \mathcal{M}_k\{\Delta_k\} - 1
\end{aligned}$$

for any k. Thus

$$\begin{aligned}
\mathcal{M}\{\Lambda \cup \Delta\} &\le 1 - \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k \cap \Delta_k\} \\
&\le 1 - \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} + 1 - \min_{1 \le k < \infty} \mathcal{M}_k\{\Delta_k\} \\
&\le \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\} + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\mathcal{M}\{\Lambda \cup \Delta\} \le \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\}.$$

Case 2: Assume $\mathcal{M}\{\Lambda\} \ge 0.5$ and $\mathcal{M}\{\Delta\} < 0.5$. When $\mathcal{M}\{\Lambda \cup \Delta\} = 0.5$, the subadditivity is obvious. Now we consider the case $\mathcal{M}\{\Lambda \cup \Delta\} > 0.5$, i.e., $\mathcal{M}\{\Lambda^c \cap \Delta^c\} < 0.5$. By using $\Lambda^c \cup \Delta = (\Lambda^c \cap \Delta^c) \cup \Delta$ and Case 1, we get

$$\mathcal{M}\{\Lambda^c \cup \Delta\} \le \mathcal{M}\{\Lambda^c \cap \Delta^c\} + \mathcal{M}\{\Delta\}.$$

Thus

$$\begin{aligned}
\mathcal{M}\{\Lambda \cup \Delta\} &= 1 - \mathcal{M}\{\Lambda^c \cap \Delta^c\} \le 1 - \mathcal{M}\{\Lambda^c \cup \Delta\} + \mathcal{M}\{\Delta\} \\
&\le 1 - \mathcal{M}\{\Lambda^c\} + \mathcal{M}\{\Delta\} = \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\}.
\end{aligned}$$

Case 3: If both $\mathcal{M}\{\Lambda\} \ge 0.5$ and $\mathcal{M}\{\Delta\} \ge 0.5$, then the subadditivity is obvious because $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\} \ge 1$. The theorem is proved.

Definition 1.9 *Assume* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ *are uncertainty spaces for* $k = 1, 2, \cdots$ *Let* $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots$, $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots$ *and* $\mathcal{M} = \mathcal{M}_1 \wedge \mathcal{M}_2 \wedge \cdots$ *Then the triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called a product uncertainty space.*

1.5 Independence

Definition 1.10 *(Liu [83]) The events* $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ *are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^*\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\} \tag{1.40}$$

where Λ_i^* *are arbitrarily chosen from* $\{\Lambda_i, \Lambda_i^c, \Gamma\}$, $i = 1, 2, \cdots, n$, *respectively, and* Γ *is the sure event.*

Remark 1.11: Especially, two events Λ_1 and Λ_2 are independent if and only if

$$\mathcal{M}\{\Lambda_1^* \cap \Lambda_2^*\} = \mathcal{M}\{\Lambda_1^*\} \wedge \mathcal{M}\{\Lambda_2^*\} \tag{1.41}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c\}$, $i = 1, 2$, respectively. That is, the following four equations hold:

$$\begin{aligned}
\mathcal{M}\{\Lambda_1 \cap \Lambda_2\} &= \mathcal{M}\{\Lambda_1\} \wedge \mathcal{M}\{\Lambda_2\},\\
\mathcal{M}\{\Lambda_1^c \cap \Lambda_2\} &= \mathcal{M}\{\Lambda_1^c\} \wedge \mathcal{M}\{\Lambda_2\},\\
\mathcal{M}\{\Lambda_1 \cap \Lambda_2^c\} &= \mathcal{M}\{\Lambda_1\} \wedge \mathcal{M}\{\Lambda_2^c\},\\
\mathcal{M}\{\Lambda_1^c \cap \Lambda_2^c\} &= \mathcal{M}\{\Lambda_1^c\} \wedge \mathcal{M}\{\Lambda_2^c\}.
\end{aligned}$$

Example 1.13: The impossible event $\emptyset$ is independent of any event Λ because the following four equations hold:

$$\begin{aligned}
\mathcal{M}\{\emptyset \cap \Lambda\} = \mathcal{M}\{\emptyset\} &= \mathcal{M}\{\emptyset\} \wedge \mathcal{M}\{\Lambda\},\\
\mathcal{M}\{\emptyset^c \cap \Lambda\} = \mathcal{M}\{\Lambda\} &= \mathcal{M}\{\emptyset^c\} \wedge \mathcal{M}\{\Lambda\},\\
\mathcal{M}\{\emptyset \cap \Lambda^c\} = \mathcal{M}\{\emptyset\} &= \mathcal{M}\{\emptyset\} \wedge \mathcal{M}\{\Lambda^c\},\\
\mathcal{M}\{\emptyset^c \cap \Lambda^c\} = \mathcal{M}\{\Lambda^c\} &= \mathcal{M}\{\emptyset^c\} \wedge \mathcal{M}\{\Lambda^c\}.
\end{aligned}$$

Example 1.14: The sure event Γ is independent of any event Λ because the following four equations hold:

$$\begin{aligned}
\mathcal{M}\{\Gamma \cap \Lambda\} = \mathcal{M}\{\Lambda\} &= \mathcal{M}\{\Gamma\} \wedge \mathcal{M}\{\Lambda\},\\
\mathcal{M}\{\Gamma^c \cap \Lambda\} = \mathcal{M}\{\Gamma^c\} &= \mathcal{M}\{\Gamma^c\} \wedge \mathcal{M}\{\Lambda\},\\
\mathcal{M}\{\Gamma \cap \Lambda^c\} = \mathcal{M}\{\Lambda^c\} &= \mathcal{M}\{\Gamma\} \wedge \mathcal{M}\{\Lambda^c\},\\
\mathcal{M}\{\Gamma^c \cap \Lambda^c\} = \mathcal{M}\{\Gamma^c\} &= \mathcal{M}\{\Gamma^c\} \wedge \mathcal{M}\{\Lambda^c\}.
\end{aligned}$$

Example 1.15: Generally speaking, an event Λ is not independent of itself because

$$\mathcal{M}\{\Lambda \cap \Lambda^c\} \neq \mathcal{M}\{\Lambda\} \wedge \mathcal{M}\{\Lambda^c\}$$

whenever $\mathcal{M}\{\Lambda\}$ is neither 1 nor 0.

Theorem 1.7 *(Liu [83]) The events $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^n \Lambda_i^*\right\} = \bigvee_{i=1}^n \mathcal{M}\{\Lambda_i^*\} \tag{1.42}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \emptyset\}$, $i = 1, 2, \cdots, n$, respectively, and $\emptyset$ is the impossible event.

Proof: Assume $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are independent events. It follows from the duality of uncertain measure that

$$\mathcal{M}\left\{\bigcup_{i=1}^n \Lambda_i^*\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^n \Lambda_i^{*c}\right\} = 1 - \bigwedge_{i=1}^n \mathcal{M}\{\Lambda_i^{*c}\} = \bigvee_{i=1}^n \mathcal{M}\{\Lambda_i^*\}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \emptyset\}$, $i = 1, 2, \cdots, n$, respectively. The equation (1.42) is proved. Conversely, if the equation (1.42) holds, then

$$\mathcal{M}\left\{\bigcap_{i=1}^n \Lambda_i^*\right\} = 1 - \mathcal{M}\left\{\bigcup_{i=1}^n \Lambda_i^{*c}\right\} = 1 - \bigvee_{i=1}^n \mathcal{M}\{\Lambda_i^{*c}\} = \bigwedge_{i=1}^n \mathcal{M}\{\Lambda_i^*\}.$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \Gamma\}$, $i = 1, 2, \cdots, n$, respectively. The equation (1.40) is true. The theorem is proved.

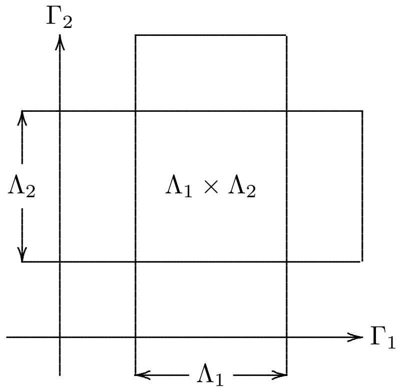


Figure 1.2: $(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2) = \Lambda_1 \times \Lambda_2$

Theorem 1.8 *(Liu [91]) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces and $\Lambda_k \in \mathcal{L}_k$ for $k = 1, 2, \cdots, n$. Then the events*

$$\Gamma_1 \times \cdots \times \Gamma_{k-1} \times \Lambda_k \times \Gamma_{k+1} \times \cdots \times \Gamma_n, \quad k = 1, 2, \cdots, n \tag{1.43}$$

are always independent in the product uncertainty space. That is, the events

$$\Lambda_1, \Lambda_2, \cdots, \Lambda_n \tag{1.44}$$

are always independent if they are from different uncertainty spaces.

Proof: For simplicity, we only prove the case of $n = 2$. It follows from the product axiom that the product uncertain measure of the intersection is

$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)\} = \mathcal{M}\{\Lambda_1 \times \Lambda_2\} = \mathcal{M}_1\{\Lambda_1\} \wedge \mathcal{M}_2\{\Lambda_2\}.$$

By using $\mathcal{M}\{\Lambda_1 \times \Gamma_2\} = \mathcal{M}_1\{\Lambda_1\}$ and $\mathcal{M}\{\Gamma_1 \times \Lambda_2\} = \mathcal{M}_2\{\Lambda_2\}$, we obtain

$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)\} = \mathcal{M}\{\Lambda_1 \times \Gamma_2\} \wedge \mathcal{M}\{\Gamma_1 \times \Lambda_2\}.$$

Similarly, we may prove that

$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c \cap (\Gamma_1 \times \Lambda_2)\} = \mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c\} \wedge \mathcal{M}\{\Gamma_1 \times \Lambda_2\},$$
$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)^c\} = \mathcal{M}\{\Lambda_1 \times \Gamma_2\} \wedge \mathcal{M}\{(\Gamma_1 \times \Lambda_2)^c\},$$
$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c \cap (\Gamma_1 \times \Lambda_2)^c\} = \mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c\} \wedge \mathcal{M}\{(\Gamma_1 \times \Lambda_2)^c\}.$$

Thus $\Lambda_1 \times \Gamma_2$ and $\Gamma_1 \times \Lambda_2$ are independent events. Furthermore, since Λ_1 and Λ_2 are understood as $\Lambda_1 \times \Gamma_2$ and $\Gamma_1 \times \Lambda_2$ in the product uncertainty space, respectively, the two events Λ_1 and Λ_2 are also independent.

1.6 Polyrectangular Theorem

Definition 1.11 *(Liu [91]) Let $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ be two uncertainty spaces. A set on $\Gamma_1 \times \Gamma_2$ is called a polyrectangle if it has the form*

$$\Lambda = \bigcup_{i=1}^{m} (\Lambda_{1i} \times \Lambda_{2i}) \tag{1.45}$$

where $\Lambda_{1i} \in \mathcal{L}_1$ and $\Lambda_{2i} \in \mathcal{L}_2$ for $i = 1, 2, \cdots, m$, and

$$\Lambda_{11} \subset \Lambda_{12} \subset \cdots \subset \Lambda_{1m}, \tag{1.46}$$

$$\Lambda_{21} \supset \Lambda_{22} \supset \cdots \supset \Lambda_{2m}. \tag{1.47}$$

A rectangle $\Lambda_1 \times \Lambda_2$ is clearly a polyrectangle. In addition, a "cross"-like set is also a polyrectangle. See Figure 1.3.

Theorem 1.9 *(Liu [91], Polyrectangular Theorem) Let $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ be two uncertainty spaces. Then the polyrectangle*

$$\Lambda = \bigcup_{i=1}^{m} (\Lambda_{1i} \times \Lambda_{2i}) \tag{1.48}$$

on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ has an uncertain measure

$$\mathcal{M}\{\Lambda\} = \bigvee_{i=1}^{m} \mathcal{M}\{\Lambda_{1i} \times \Lambda_{2i}\}. \tag{1.49}$$

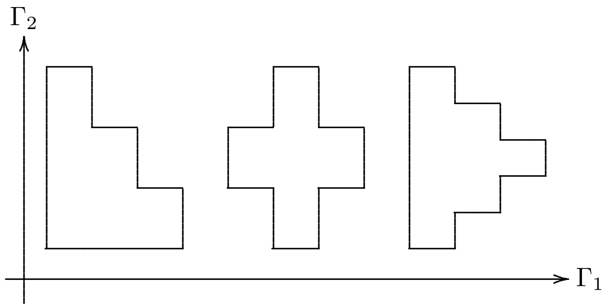


Figure 1.3: Three Polyrectangles

Proof: It is clear that the maximum rectangle in the polyrectangle Λ is one of $\Lambda_{1i} \times \Lambda_{2i}$, $i = 1, 2, \cdots, n$. Denote the maximum rectangle by $\Lambda_{1k} \times \Lambda_{2k}$. Case I: If

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_1\{\Lambda_{1k}\},$$

then the maximum rectangle in Λ^c is $\Lambda_{1k}^c \times \Lambda_{2,k+1}^c$, and

$$\mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = \mathcal{M}_1\{\Lambda_{1k}^c\} = 1 - \mathcal{M}_1\{\Lambda_{1k}\}.$$

Thus

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} + \mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = 1.$$

Case II: If

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_2\{\Lambda_{2k}\},$$

then the maximum rectangle in Λ^c is $\Lambda_{1,k-1}^c \times \Lambda_{2k}^c$, and

$$\mathcal{M}\{\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\} = \mathcal{M}_2\{\Lambda_{2k}^c\} = 1 - \mathcal{M}_2\{\Lambda_{2k}\}.$$

Thus

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} + \mathcal{M}\{\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\} = 1.$$

No matter what case happens, the sum of the uncertain measures of the maximum rectangles in Λ and Λ^c is always 1. It follows from the product axiom that (1.49) holds.

Remark 1.12: Since $\mathcal{M}\{\Lambda_{1i} \times \Lambda_{2i}\} = \mathcal{M}_1\{\Lambda_{1i}\} \wedge \mathcal{M}_2\{\Lambda_{2i}\}$ for each index i, we also have

$$\mathcal{M}\{\Lambda\} = \bigvee_{i=1}^{m} \mathcal{M}_1\{\Lambda_{1i}\} \wedge \mathcal{M}_2\{\Lambda_{2i}\}. \tag{1.50}$$

Remark 1.13: Note that the polyrectangular theorem is also applicable to the polyrectangles that are unions of infinitely many rectangles. In this case, the polyrectangles may become the shapes in Figure 1.4.

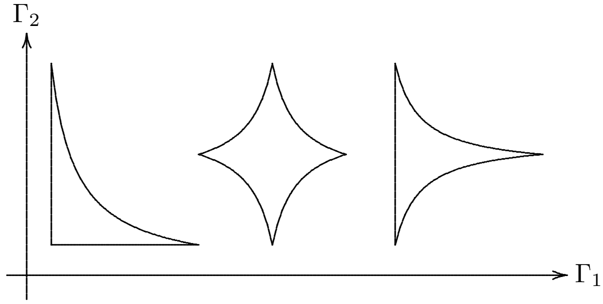


Figure 1.4: Three Deformed Polyrectangles

1.7 Conditional Uncertain Measure

We consider the uncertain measure of an event Λ after it has been learned that some other event A has occurred. This new uncertain measure of Λ is called the *conditional uncertain measure* of Λ given A.

In order to define a conditional uncertain measure $\mathcal{M}\{\Lambda|A\}$, at first we have to enlarge $\mathcal{M}\{\Lambda \cap A\}$ because $\mathcal{M}\{\Lambda \cap A\} < 1$ for all events whenever $\mathcal{M}\{A\} < 1$. It seems that we have no alternative but to divide $\mathcal{M}\{\Lambda \cap A\}$ by $\mathcal{M}\{A\}$. Unfortunately, $\mathcal{M}\{\Lambda \cap A\}/\mathcal{M}\{A\}$ is not always an uncertain measure. However, the value $\mathcal{M}\{\Lambda|A\}$ should not be greater than $\mathcal{M}\{\Lambda \cap A\}/\mathcal{M}\{A\}$ (otherwise the normality will be lost), i.e.,

$$\mathcal{M}\{\Lambda|A\} \le \frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}}. \tag{1.51}$$

On the other hand, in order to preserve the duality, we should have

$$\mathcal{M}\{\Lambda|A\} = 1 - \mathcal{M}\{\Lambda^c|A\} \ge 1 - \frac{\mathcal{M}\{\Lambda^c \cap A\}}{\mathcal{M}\{A\}}. \tag{1.52}$$

Furthermore, since $(\Lambda \cap A) \cup (\Lambda^c \cap A) = A$, we have $\mathcal{M}\{A\} \le \mathcal{M}\{\Lambda \cap A\} + \mathcal{M}\{\Lambda^c \cap A\}$ by using the subadditivity axiom. Thus

$$0 \le 1 - \frac{\mathcal{M}\{\Lambda^c \cap A\}}{\mathcal{M}\{A\}} \le \frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}} \le 1. \tag{1.53}$$

Hence any numbers between $1-\mathcal{M}\{\Lambda^c \cap A\}/\mathcal{M}\{A\}$ and $\mathcal{M}\{\Lambda \cap A\}/\mathcal{M}\{A\}$ are reasonable values that the conditional uncertain measure may take. Based on the maximum uncertainty principle (Liu [76]), we have the following conditional uncertain measure.

Definition 1.12 *(Liu [76]) Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, and* $\Lambda, A \in$

$\mathcal{L}$. Then the conditional uncertain measure of Λ given A is defined by

$$\mathcal{M}\{\Lambda|A\} = \begin{cases} \dfrac{\mathcal{M}\{\Lambda\cap A\}}{\mathcal{M}\{A\}}, & \text{if } \dfrac{\mathcal{M}\{\Lambda\cap A\}}{\mathcal{M}\{A\}} < 0.5 \\ 1-\dfrac{\mathcal{M}\{\Lambda^c\cap A\}}{\mathcal{M}\{A\}}, & \text{if } \dfrac{\mathcal{M}\{\Lambda^c\cap A\}}{\mathcal{M}\{A\}} < 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{1.54}$$

provided that $\mathcal{M}\{A\} > 0$.

Remark 1.14: It follows immediately from the definition of conditional uncertain measure that

$$1-\frac{\mathcal{M}\{\Lambda^c\cap A\}}{\mathcal{M}\{A\}} \le \mathcal{M}\{\Lambda|A\} \le \frac{\mathcal{M}\{\Lambda\cap A\}}{\mathcal{M}\{A\}}. \tag{1.55}$$

Remark 1.15: The conditional uncertain measure $\mathcal{M}\{\Lambda|A\}$ yields the posterior uncertain measure of Λ after the occurrence of event A.

Theorem 1.10 *(Liu [76]) Let $(\Gamma,\mathcal{L},\mathcal{M})$ be an uncertainty space, and let A be an event with $\mathcal{M}\{A\} > 0$. Then $\mathcal{M}\{\cdot|A\}$ defined by (1.54) is an uncertain measure, and $(\Gamma,\mathcal{L},\mathcal{M}\{\cdot|A\})$ is an uncertainty space.*

Proof: It is sufficient to prove that $\mathcal{M}\{\cdot|A\}$ satisfies the normality, duality and subadditivity axioms. At first, it satisfies the normality axiom, i.e.,

$$\mathcal{M}\{\Gamma|A\} = 1-\frac{\mathcal{M}\{\Gamma^c\cap A\}}{\mathcal{M}\{A\}} = 1-\frac{\mathcal{M}\{\emptyset\}}{\mathcal{M}\{A\}} = 1.$$

For any event Λ, if

$$\frac{\mathcal{M}\{\Lambda\cap A\}}{\mathcal{M}\{A\}} \ge 0.5, \quad \frac{\mathcal{M}\{\Lambda^c\cap A\}}{\mathcal{M}\{A\}} \ge 0.5,$$

then we have $\mathcal{M}\{\Lambda|A\}+\mathcal{M}\{\Lambda^c|A\} = 0.5+0.5 = 1$ immediately. Otherwise, without loss of generality, suppose

$$\frac{\mathcal{M}\{\Lambda\cap A\}}{\mathcal{M}\{A\}} < 0.5 < \frac{\mathcal{M}\{\Lambda^c\cap A\}}{\mathcal{M}\{A\}},$$

then we have

$$\mathcal{M}\{\Lambda|A\}+\mathcal{M}\{\Lambda^c|A\} = \frac{\mathcal{M}\{\Lambda\cap A\}}{\mathcal{M}\{A\}} + \left(1-\frac{\mathcal{M}\{\Lambda\cap A\}}{\mathcal{M}\{A\}}\right) = 1.$$

That is, $\mathcal{M}\{\cdot|A\}$ satisfies the duality axiom. Finally, for any countable sequence $\{\Lambda_i\}$ of events, if $\mathcal{M}\{\Lambda_i|A\} < 0.5$ for all i, it follows from (1.55) and the subadditivity axiom that

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i \mid A\right\} \le \frac{\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i \cap A\right\}}{\mathcal{M}\{A\}} \le \frac{\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i \cap A\}}{\mathcal{M}\{A\}} = \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i|A\}.$$

Suppose there is one term greater than 0.5, say

$$\mathcal{M}\{\Lambda_1|A\} \ge 0.5, \quad \mathcal{M}\{\Lambda_i|A\} < 0.5, \quad i = 2, 3, \cdots$$

If $\mathcal{M}\{\cup_i\Lambda_i|A\} = 0.5$, then we immediately have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i \mid A\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i|A\}.$$

If $\mathcal{M}\{\cup_i\Lambda_i|A\} > 0.5$, we may prove the above inequality by the following facts:

$$\Lambda_1^c \cap A \subset \bigcup_{i=2}^{\infty}(\Lambda_i \cap A) \cup \left(\bigcap_{i=1}^{\infty}\Lambda_i^c \cap A\right),$$

$$\mathcal{M}\{\Lambda_1^c \cap A\} \le \sum_{i=2}^{\infty}\mathcal{M}\{\Lambda_i \cap A\} + \mathcal{M}\left\{\bigcap_{i=1}^{\infty}\Lambda_i^c \cap A\right\},$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i \mid A\right\} = 1 - \frac{\mathcal{M}\left\{\bigcap_{i=1}^{\infty}\Lambda_i^c \cap A\right\}}{\mathcal{M}\{A\}},$$

$$\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i|A\} \ge 1 - \frac{\mathcal{M}\{\Lambda_1^c \cap A\}}{\mathcal{M}\{A\}} + \frac{\sum_{i=2}^{\infty}\mathcal{M}\{\Lambda_i \cap A\}}{\mathcal{M}\{A\}}.$$

If there are at least two terms greater than 0.5, then the subadditivity is clearly true. Thus $\mathcal{M}\{\cdot|A\}$ satisfies the subadditivity axiom. Hence $\mathcal{M}\{\cdot|A\}$ is an uncertain measure. Furthermore, $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|A\})$ is an uncertainty space.

1.8 Bibliographic Notes

When no samples are available to estimate a probability distribution, we have to invite some domain experts to evaluate the belief degree that each event will happen. Perhaps some people think that the belief degree is subjective probability or fuzzy concept. However, Liu [85] declared that it is usually inappropriate because both probability theory and fuzzy set theory may lead to counterintuitive results in this case.

In order to rationally deal with belief degrees, uncertainty theory was founded by Liu [76] in 2007 and perfected by Liu [79] in 2009. The core of uncertainty theory is uncertain measure defined by the normality axiom, duality axiom, subadditivity axiom, and product axiom. In practice, uncertain measure is interpreted as the personal belief degree of an uncertain event that may happen.

Uncertain measure was also actively investigated by Gao [40], Liu [83], Zhang [199], Peng-Iwamura [121], and Liu [91], among others. Since then, the tool of uncertain measure was well developed and became a rigorous footstone of uncertainty theory.

Chapter 2

Uncertain Variable

Uncertain variable is a fundamental concept in uncertainty theory. It is used to represent quantities with uncertainty. The emphasis in this chapter is mainly on uncertain variable, uncertainty distribution, independence, operational law, expected value, variance, moments, distance, entropy, conditional uncertainty distribution, uncertain sequence, uncertain vector, and uncertain matrix.

2.1 Uncertain Variable

Roughly speaking, an uncertain variable is a measurable function on an uncertainty space. A formal definition is given as follows.

Definition 2.1 *(Liu [76]) An uncertain variable is a function* ξ *from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers such that* $\{\xi \in B\}$ *is an event for any Borel set* B *of real numbers.*

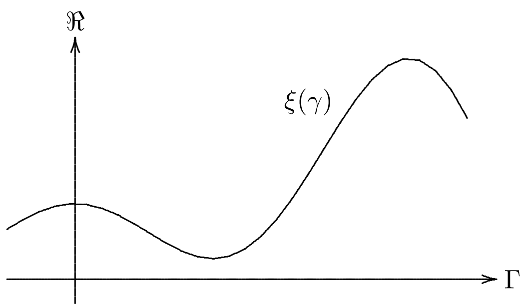


Figure 2.1: An Uncertain Variable

Remark 2.1: Note that the event $\{\xi \in B\}$ is a subset of the universal set Γ, i.e.,

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}. \tag{2.1}$$

Example 2.1: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.4$. Then

$$\xi(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases} \tag{2.2}$$

is an uncertain variable. Furthermore, we have

$$\mathcal{M}\{\xi = 0\} = \mathcal{M}\{\gamma \mid \xi(\gamma) = 0\} = \mathcal{M}\{\gamma_1\} = 0.6, \tag{2.3}$$

$$\mathcal{M}\{\xi = 1\} = \mathcal{M}\{\gamma \mid \xi(\gamma) = 1\} = \mathcal{M}\{\gamma_2\} = 0.4. \tag{2.4}$$

Example 2.2: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then

$$\xi(\gamma) = 3\gamma, \quad \forall \gamma \in \Gamma \tag{2.5}$$

is an uncertain variable. Furthermore, we have

$$\mathcal{M}\{\xi = 1\} = \mathcal{M}\{\gamma \mid \xi(\gamma) = 1\} = \mathcal{M}\{1/3\} = 0, \tag{2.6}$$

$$\mathcal{M}\{\xi \in [0, 2]\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \in [0, 2]\} = \mathcal{M}\{[0, 2/3]\} = 2/3, \tag{2.7}$$

$$\mathcal{M}\{\xi > 2\} = \mathcal{M}\{\gamma \mid \xi(\gamma) > 2\} = \mathcal{M}\{(2/3, 1]\} = 1/3. \tag{2.8}$$

Example 2.3: A real number c may be regarded as a special uncertain variable. In fact, it is the constant function

$$\xi(\gamma) \equiv c \tag{2.9}$$

on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Furthermore, for any Borel set B of real numbers, we have

$$\mathcal{M}\{\xi \in B\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \in B\} = \mathcal{M}\{\Gamma\} = 1, \quad \text{if } c \in B, \tag{2.10}$$

$$\mathcal{M}\{\xi \in B\} = \mathcal{M}\{\gamma \mid \xi(\gamma) \in B\} = \mathcal{M}\{\emptyset\} = 0, \quad \text{if } c \notin B. \tag{2.11}$$

Example 2.4: Let ξ be an uncertain variable and let b be a real number. Then

$$\{\xi = b\}^c = \{\gamma \mid \xi(\gamma) = b\}^c = \{\gamma \mid \xi(\gamma) \neq b\} = \{\xi \neq b\}.$$

Thus $\{\xi = b\}$ and $\{\xi \neq b\}$ are opposite events. Furthermore, by the duality axiom, we obtain

$$\mathcal{M}\{\xi = b\} + \mathcal{M}\{\xi \neq b\} = 1. \tag{2.12}$$

Exercise 2.1: Let ξ be an uncertain variable and let B be a Borel set of real numbers. Show that $\{\xi \in B\}$ and $\{\xi \in B^c\}$ are opposite events, and

$$\mathcal{M}\{\xi \in B\} + \mathcal{M}\{\xi \in B^c\} = 1. \tag{2.13}$$

Exercise 2.2: Let ξ and η be two uncertain variables. Show that $\{\xi \geq \eta\}$ and $\{\xi < \eta\}$ are opposite events, and

$$\mathcal{M}\{\xi \geq \eta\} + \mathcal{M}\{\xi < \eta\} = 1. \tag{2.14}$$

Definition 2.2 *An uncertain variable ξ on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be (a) nonnegative if $\mathcal{M}\{\xi < 0\} = 0$; and (b) positive if $\mathcal{M}\{\xi \leq 0\} = 0$.*

Definition 2.3 *Let ξ and η be uncertain variables defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We say $\xi = \eta$ if $\xi(\gamma) = \eta(\gamma)$ for almost all $\gamma \in \Gamma$.*

Definition 2.4 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables, and let f be a real-valued measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable defined by*

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \cdots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma. \tag{2.15}$$

Example 2.5: Let ξ_1 and ξ_2 be two uncertain variables. Then the sum $\xi = \xi_1 + \xi_2$ is an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma) + \xi_2(\gamma), \quad \forall \gamma \in \Gamma.$$

The product $\xi = \xi_1 \xi_2$ is also an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma) \cdot \xi_2(\gamma), \quad \forall \gamma \in \Gamma.$$

The reader may wonder whether $\xi(\gamma)$ defined by (2.15) is an uncertain variable. The following theorem answers this question.

Theorem 2.1 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables, and let f be a real-valued measurable function. Then $f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable.*

Proof: Since $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables, they are measurable functions from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. Thus $f(\xi_1, \xi_2, \cdots, \xi_n)$ is also a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. Hence $f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable.

2.2 Uncertainty Distribution

This section introduces a concept of uncertainty distribution in order to describe uncertain variables. Mention that uncertainty distribution is a carrier of incomplete information of uncertain variable. However, in many cases, it is sufficient to know the uncertainty distribution rather than the uncertain variable itself.

Definition 2.5 *(Liu [76]) The uncertainty distribution* Φ *of an uncertain variable* ξ *is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \le x\} \tag{2.16}$$

for any real number x*.*

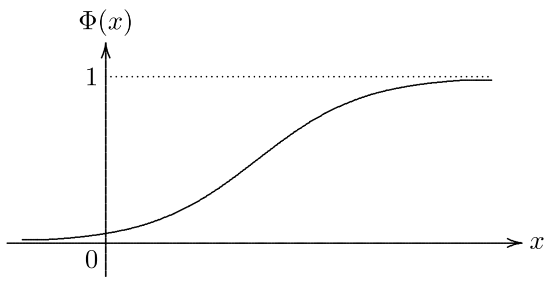


Figure 2.2: An Uncertainty Distribution

Exercise 2.3: A real number c is a special uncertain variable $\xi(\gamma) \equiv c$. Show that such an uncertain variable has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x \ge c. \end{cases}$$

Exercise 2.4: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.7$, $\mathcal{M}\{\gamma_2\} = 0.3$. Show that the uncertain variable

$$\xi(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases}$$

has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.7, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

Exercise 2.5: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$. Show that the uncertain variable

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 2, & \text{if } \gamma = \gamma_2 \\ 3, & \text{if } \gamma = \gamma_3 \end{cases}$$

has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 1 \\ 0.6, & \text{if } 1 \le x < 2 \\ 0.8, & \text{if } 2 \le x < 3 \\ 1, & \text{if } x \ge 3. \end{cases}$$

Exercise 2.6: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. (i) Show that the uncertain variable

$$\xi(\gamma) = \gamma, \quad \forall \gamma \in [0, 1] \tag{2.17}$$

has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x > 1. \end{cases} \tag{2.18}$$

(ii) What is the uncertainty distribution of $\xi(\gamma) = 1 - \gamma$? (iii) What do those two uncertain variables make you think about? (iv) Design a third uncertain variable whose uncertainty distribution is also (2.18).

Exercise 2.7: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. (i) Show that the uncertain variable $\xi(\gamma) = \gamma^2$ has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x > 1. \end{cases} \tag{2.19}$$

(ii) What is the uncertainty distribution of $\xi(\gamma) = \sqrt{\gamma}$? (iii) What is the uncertainty distribution of $\xi(\gamma) = 1/\gamma$?

Definition 2.6 *Uncertain variables are said to be identically distributed if they have the same uncertainty distribution.*

It is clear that uncertain variables ξ and η are identically distributed if $\xi = \eta$. However, identical distribution does not imply $\xi = \eta$. For example, let $(\Gamma, \mathcal{L}, \mathcal{M})$ be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Define

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ -1, & \text{if } \gamma = \gamma_2, \end{cases} \qquad \eta(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

Then ξ and η have the same uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x < -1 \\ 0.5, & \text{if } -1 \le x < 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

Thus the two uncertain variables ξ and η are identically distributed but $\xi \ne \eta$.

What is a "completely unknown number"?

A "completely unknown number" may be regarded as an uncertain variable whose uncertainty distribution is

$$\Phi(x) = 0.5 \tag{2.20}$$

for any real number x.

How old is John?

Someone thinks John is neither younger than 24 nor older than 28, and presents an uncertainty distribution of John's age as follows,

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 24 \\ (x-24)/4, & \text{if } 24 \le x \le 28 \\ 1, & \text{if } x \ge 28. \end{cases} \tag{2.21}$$

How tall is James?

Someone thinks James' height is between 180 and 185 centimeters, and presents the following uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x-180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } x \ge 185. \end{cases} \tag{2.22}$$

Sufficient and Necessary Condition

Theorem 2.2 *(Peng-Iwamura Theorem [120]) A function $\Phi(x) : \Re \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.*

Proof: It is obvious that an uncertainty distribution Φ is a monotone increasing function. In addition, both $\Phi(x) \not\equiv 0$ and $\Phi(x) \not\equiv 1$ follow from the asymptotic theorem immediately. Conversely, suppose that Φ is a monotone increasing function but $\Phi(x) \not\equiv 0$ and $\Phi(x) \not\equiv 1$. We will prove that there is an uncertain variable whose uncertainty distribution is just Φ. Let $\mathcal{C}$ be a collection of all intervals of the form $(-\infty, a]$, (b, ∞), $\emptyset$ and $\Re$. We define a set function on $\Re$ as follows,

$$\mathcal{M}\{(-\infty, a]\} = \Phi(a),$$
$$\mathcal{M}\{(b, +\infty)\} = 1 - \Phi(b),$$
$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Re\} = 1.$$

For an arbitrary Borel set B of real numbers, there exists a sequence $\{A_i\}$ in $\mathcal{C}$ such that

$$B \subset \bigcup_{i=1}^{\infty} A_i.$$

Note that such a sequence is not unique. We define a set function $\mathcal{M}\{B\}$ by

$$\mathcal{M}\{B\} = \begin{cases} \inf\limits_{B \subset \bigcup_{i=1}^{\infty} A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{B \subset \bigcup_{i=1}^{\infty} A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 1 - \inf\limits_{B^c \subset \bigcup_{i=1}^{\infty} A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{B^c \subset \bigcup_{i=1}^{\infty} A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Then the set function $\mathcal{M}$ is indeed an uncertain measure on $\Re$, and the uncertain variable defined by the identity function $\xi(\gamma) = \gamma$ has the uncertainty distribution Φ.

Example 2.6: It follows from the sufficient and necessary condition that the function

$$\Phi(x) \equiv 0.5 \tag{2.23}$$

is an uncertainty distribution. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\Re$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Re \\ 0.5, & \text{otherwise.} \end{cases} \tag{2.24}$$

Then the uncertain variable $\xi(\gamma) = \gamma$ has the uncertainty distribution (2.23).

Exercise 2.8: (i) Design an uncertain variable whose uncertainty distribution is

$$\Phi(x) = 0.4 \tag{2.25}$$

for any real number x. (ii) Design an uncertain variable whose uncertainty distribution is

$$\Phi(x) = 0.6 \tag{2.26}$$

for any real number x.

Exercise 2.9: Design an uncertain variable whose uncertainty distribution is

$$\Phi(x) = (1 + \exp(-x))^{-1} \tag{2.27}$$

for any real number x.

Some Uncertainty Distributions

Definition 2.7 *An uncertain variable ξ is called linear if it has a linear uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x \ge b \end{cases} \tag{2.28}$$

denoted by $\mathcal{L}(a, b)$ where a and b are real numbers with $a < b$.

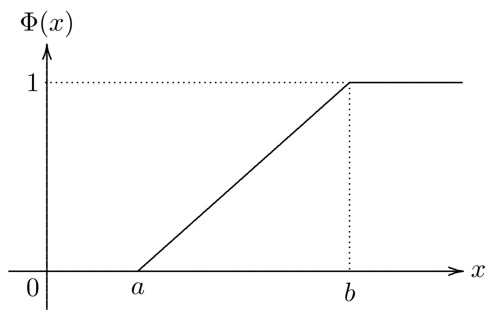


Figure 2.3: Linear Uncertainty Distribution

Example 2.7: John's age (2.21) is a linear uncertain variable $\mathcal{L}(24, 28)$, and James' height (2.22) is another linear uncertain variable $\mathcal{L}(180, 185)$.

Definition 2.8 *An uncertain variable ξ is called zigzag if it has a zigzag* *uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ \dfrac{x-a}{2(b-a)}, & \text{if } a \le x \le b \\ \dfrac{x+c-2b}{2(c-b)}, & \text{if } b \le x \le c \\ 1, & \text{if } x \ge c \end{cases} \tag{2.29}$$

denoted by $\mathcal{Z}(a,b,c)$ *where* a,b,c *are real numbers with* $a<b<c$*.*

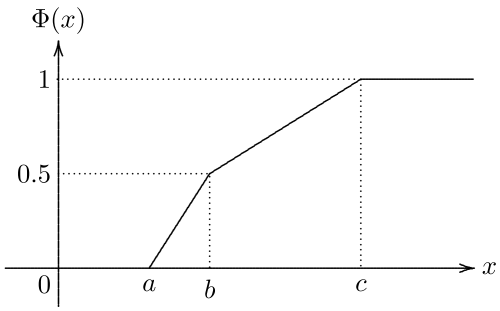


Figure 2.4: Zigzag Uncertainty Distribution

Definition 2.9 *An uncertain variable* ξ *is called normal if it has a normal uncertainty distribution*

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re \tag{2.30}$$

denoted by $\mathcal{N}(e,\sigma)$ *where* e *and* σ *are real numbers with* $\sigma > 0$*.*

Definition 2.10 *An uncertain variable* ξ *is called lognormal if* $\ln\xi$ *is a normal uncertain variable* $\mathcal{N}(e,\sigma)$*. In other words, a lognormal uncertain variable has an uncertainty distribution*

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-\ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \ge 0 \tag{2.31}$$

denoted by $\mathcal{LOGN}(e,\sigma)$*, where* e *and* σ *are real numbers with* $\sigma > 0$*.*

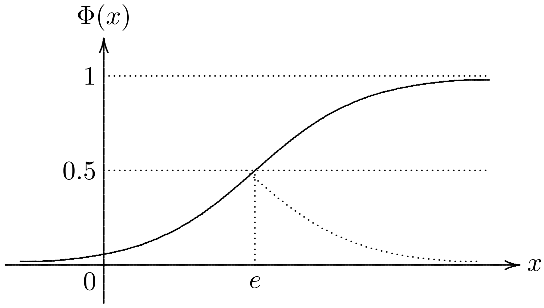


Figure 2.5: Normal Uncertainty Distribution

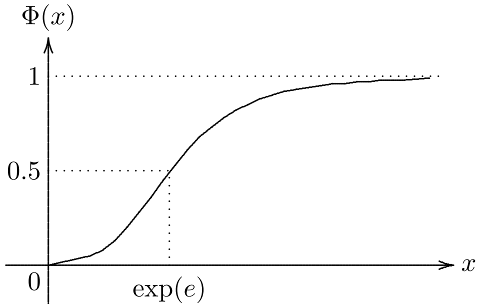


Figure 2.6: Lognormal Uncertainty Distribution

Definition 2.11 *An uncertain variable ξ is called empirical if it has an empirical uncertainty distribution*

$$\Phi(x)=\begin{cases} 0, & \text{if } x<x_1 \\ \alpha_i+\dfrac{(\alpha_{i+1}-\alpha_i)(x-x_i)}{x_{i+1}-x_i}, & \text{if } x_i\le x\le x_{i+1},\ 1\le i<n \\ 1, & \text{if } x>x_n \end{cases} \tag{2.32}$$

where $x_1<x_2<\cdots<x_n$ and $0\le\alpha_1\le\alpha_2\le\cdots\le\alpha_n\le 1$.

Measure Inversion Theorem

Theorem 2.3 *(Liu [83], Measure Inversion Theorem) Let ξ be an uncertain variable with uncertainty distribution Φ. Then for any real number x, we have*

$$\mathcal{M}\{\xi\le x\}=\Phi(x),\quad \mathcal{M}\{\xi>x\}=1-\Phi(x). \tag{2.33}$$

Proof: The equation $\mathcal{M}\{\xi\le x\}=\Phi(x)$ follows from the definition of uncertainty distribution immediately. By using the duality of uncertain measure,

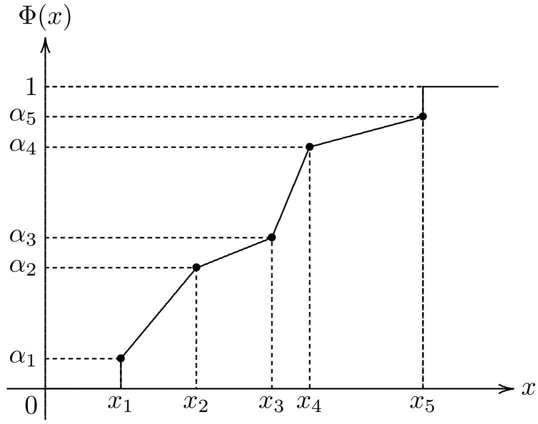


Figure 2.7: Empirical Uncertainty Distribution

we get

$$\mathcal{M}\{\xi > x\} = 1 - \mathcal{M}\{\xi \le x\} = 1 - \Phi(x).$$

The theorem is verified.

Remark 2.2: When the uncertainty distribution Φ is a continuous function, we also have

$$\mathcal{M}\{\xi < x\} = \Phi(x), \quad \mathcal{M}\{\xi \ge x\} = 1 - \Phi(x). \tag{2.34}$$

Remark 2.3: Perhaps some readers would like to get an exactly scalar value of the uncertain measure $\mathcal{M}\{a \le \xi \le b\}$. Generally speaking, it is an impossible job (except $a = -\infty$ or $b = +\infty$) if only an uncertainty distribution is available. I would like to ask if there is a need to know it. In fact, it is not necessary for practical purpose. Would you believe? I hope so!

Regular Uncertainty Distribution

Definition 2.12 *(Liu [83]) An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and*

$$\lim_{x\to-\infty} \Phi(x) = 0, \quad \lim_{x\to+\infty} \Phi(x) = 1. \tag{2.35}$$

For example, linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution, and lognormal uncertainty distribution are all regular.

Inverse Uncertainty Distribution

It is clear that a regular uncertainty distribution $\Phi(x)$ has an inverse function on the range of x with $0 < \Phi(x) < 1$, and the inverse function $\Phi^{-1}(\alpha)$ exists on the open interval $(0, 1)$.

Definition 2.13 *(Liu [83]) Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.*

Note that the inverse uncertainty distribution $\Phi^{-1}(\alpha)$ is well defined on the open interval $(0, 1)$. If needed, we may extend the domain to $[0, 1]$ via

$$\Phi^{-1}(0) = \lim_{\alpha \downarrow 0} \Phi^{-1}(\alpha), \quad \Phi^{-1}(1) = \lim_{\alpha \uparrow 1} \Phi^{-1}(\alpha). \tag{2.36}$$

Example 2.8: The inverse uncertainty distribution of linear uncertain variable $\mathcal{L}(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b. \tag{2.37}$$

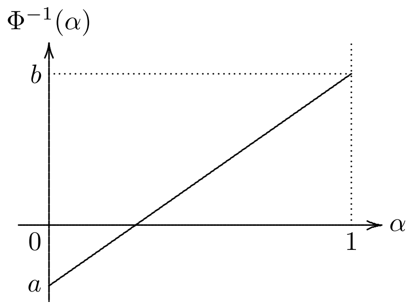


Figure 2.8: Inverse Linear Uncertainty Distribution

Example 2.9: The inverse uncertainty distribution of zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } \alpha \ge 0.5. \end{cases} \tag{2.38}$$

Example 2.10: The inverse uncertainty distribution of normal uncertain variable $\mathcal{N}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}. \tag{2.39}$$

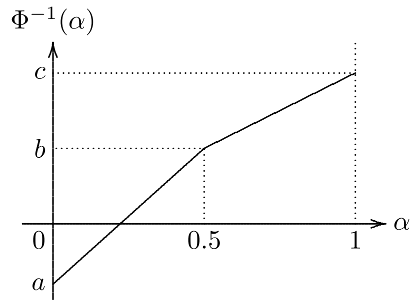


Figure 2.9: Inverse Zigzag Uncertainty Distribution

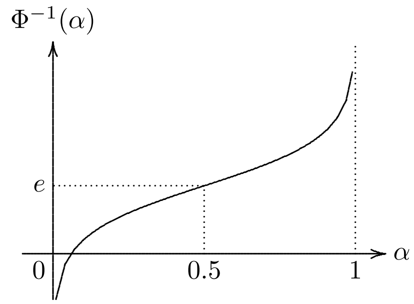


Figure 2.10: Inverse Normal Uncertainty Distribution

Example 2.11: The inverse uncertainty distribution of lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = \exp\left(e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right). \tag{2.40}$$

Theorem 2.4 *A function Φ^{-1} is an inverse uncertainty distribution of an uncertain variable ξ if and only if*

$$\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} = \alpha \tag{2.41}$$

for all $\alpha \in (0, 1)$.

Proof: Suppose Φ^{-1} is the inverse uncertainty distribution of ξ. Then for any α, we have

$$\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} = \Phi(\Phi^{-1}(\alpha)) = \alpha.$$

Conversely, suppose Φ^{-1} meets (2.41). Write $x = \Phi^{-1}(\alpha)$. Then $\alpha = \Phi(x)$ and

$$\mathcal{M}\{\xi \le x\} = \alpha = \Phi(x).$$

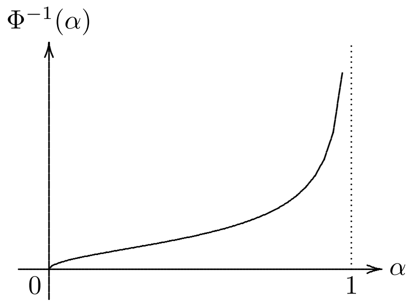


Figure 2.11: Inverse Lognormal Uncertainty Distribution

That is, Φ is the uncertainty distribution of ξ and Φ^{-1} is its inverse uncertainty distribution. The theorem is verified.

Theorem 2.5 *(Liu [88], Sufficient and Necessary Condition) A function $\Phi^{-1}(\alpha) : (0,1) \to \Re$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function with respect to α.*

Proof: Suppose $\Phi^{-1}(\alpha)$ is an inverse uncertainty distribution. It follows from the definition of inverse uncertainty distribution that $\Phi^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to $\alpha \in (0,1)$.

Conversely, suppose $\Phi^{-1}(\alpha)$ is a continuous and strictly increasing function on $(0,1)$. Define

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le \lim\limits_{\alpha\downarrow 0} \Phi^{-1}(\alpha) \\ \alpha, & \text{if } x = \Phi^{-1}(\alpha) \\ 1, & \text{if } x \ge \lim\limits_{\alpha\uparrow 1} \Phi^{-1}(\alpha). \end{cases}$$

It follows from Peng-Iwamura theorem that $\Phi(x)$ is an uncertainty distribution of some uncertain variable ξ. Then for each $\alpha \in (0,1)$, we have

$$\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} = \Phi(\Phi^{-1}(\alpha)) = \alpha.$$

Thus $\Phi^{-1}(\alpha)$ is just the inverse uncertainty distribution of the uncertain variable ξ. The theorem is verified.

2.3 Independence

Note that an uncertain variable is a measurable function from an uncertainty space to the set of real numbers. The independence of two functions means that knowing the value of one does not change our estimation of the value of another. What uncertain variables meet this condition? A typical case is that they are defined on different uncertainty spaces. For example, let $\xi_1(\gamma_1)$ and $\xi_2(\gamma_2)$ be uncertain variables on the uncertainty spaces $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$, respectively. It is clear that they are also uncertain variables on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$. Then for any Borel sets B_1 and B_2 of real numbers, we have

$$\begin{aligned}
&\mathcal{M}\{(\xi_1 \in B_1) \cap (\xi_2 \in B_2)\} \\
&= \mathcal{M}\left\{(\gamma_1, \gamma_2) \,|\, \xi_1(\gamma_1) \in B_1,\ \xi_2(\gamma_2) \in B_2\right\} \\
&= \mathcal{M}\left\{(\gamma_1 \,|\, \xi_1(\gamma_1) \in B_1) \times (\gamma_2 \,|\, \xi_2(\gamma_2) \in B_2)\right\} \\
&= \mathcal{M}_1\left\{\gamma_1 \,|\, \xi_1(\gamma_1) \in B_1\right\} \wedge \mathcal{M}_2\left\{\gamma_2 \,|\, \xi_2(\gamma_2) \in B_2\right\} \\
&= \mathcal{M}\left\{\xi_1 \in B_1\right\} \wedge \mathcal{M}\left\{\xi_2 \in B_2\right\}.
\end{aligned}$$

That is,

$$\mathcal{M}\{(\xi_1 \in B_1) \cap (\xi_2 \in B_2)\} = \mathcal{M}\left\{\xi_1 \in B_1\right\} \wedge \mathcal{M}\left\{\xi_2 \in B_2\right\}. \tag{2.42}$$

Thus we say two uncertain variables are independent if the equation (2.42) holds. Generally, we may define independence in the following form.

Definition 2.14 *(Liu [79]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{\xi_i \in B_i\right\} \tag{2.43}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Exercise 2.10: Show that a constant (a special uncertain variable) is always independent of any uncertain variable.

Exercise 2.11: John gives Tom 2 dollars. Thus John gets "-2 dollars" and Tom "+2 dollars". Are John's "-2 dollars" and Tom's "+2 dollars" independent? Why?

Exercise 2.12: Let ξ be an uncertain variable. Are ξ and $1-\xi$ independent? Please justify your answer.

Theorem 2.6 *(Liu [79]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\xi_i \in B_i\right\} \tag{2.44}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Proof: It follows from the duality of uncertain measure that $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i^c)\right\}$$
$$= 1 - \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i^c\} = \bigvee_{i=1}^{n}\mathcal{M}\left\{\xi_i \in B_i\right\}.$$

Thus the proof is complete.

Theorem 2.7 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables, and let $f_1, f_2, \cdots, f_n$ be measurable functions. Then $f_1(\xi_1), f_2(\xi_2), \cdots, f_n(\xi_n)$ are independent uncertain variables.*

Proof: For any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers, it follows from the definition of independence that

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(f_i(\xi_i) \in B_i)\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in f_i^{-1}(B_i))\right\}$$
$$= \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in f_i^{-1}(B_i)\} = \bigwedge_{i=1}^{n}\mathcal{M}\{f_i(\xi_i) \in B_i\}.$$

Thus $f_1(\xi_1), f_2(\xi_2), \cdots, f_n(\xi_n)$ are independent uncertain variables.

2.4 Operational Law: Inverse Distribution

This section provides some operational laws for calculating the inverse uncertainty distributions of strictly increasing function, strictly decreasing function, and strictly monotone function of uncertain variables.

Strictly Increasing Function of Uncertain Variables

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly increasing if

$$f(x_1, x_2, \cdots, x_n) \le f(y_1, y_2, \cdots, y_n) \tag{2.45}$$

whenever $x_i \le y_i$ for $i = 1, 2, \cdots, n$, and

$$f(x_1, x_2, \cdots, x_n) < f(y_1, y_2, \cdots, y_n) \tag{2.46}$$

whenever $x_i < y_i$ for $i = 1, 2, \cdots, n$. The following are strictly increasing functions,

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n,$$
$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n,$$
$$f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n,$$
$$f(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n, \quad x_1, x_2, \cdots, x_n \ge 0.$$

Theorem 2.8 *(Liu [83]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.47}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)). \tag{2.48}$$

Proof: For simplicity, we only prove the case $n = 2$. At first, we always have

$$\{\xi \le \Psi^{-1}(\alpha)\} \equiv \{f(\xi_1, \xi_2) \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha))\}.$$

On the one hand, since f is a strictly increasing function, we obtain

$$\{\xi \le \Psi^{-1}(\alpha)\} \supset \{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \le \Phi_2^{-1}(\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\begin{aligned}\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} &\ge \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \le \Phi_2^{-1}(\alpha)\}\\ &= \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \wedge \mathcal{M}\{\xi_2 \le \Phi_2^{-1}(\alpha)\}\\ &= \alpha \wedge \alpha = \alpha.\end{aligned}$$

On the other hand, since f is a strictly increasing function, we obtain

$$\{\xi \le \Psi^{-1}(\alpha)\} \subset \{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \le \Phi_2^{-1}(\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\begin{aligned}\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} &\le \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \le \Phi_2^{-1}(\alpha)\}\\ &= \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \vee \mathcal{M}\{\xi_2 \le \Phi_2^{-1}(\alpha)\}\\ &= \alpha \vee \alpha = \alpha.\end{aligned}$$

It follows that $\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \alpha$. That is, Ψ^{-1} is just the inverse uncertainty distribution of ξ. The theorem is proved.

Exercise 2.13: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the sum

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \tag{2.49}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha). \tag{2.50}$$

Exercise 2.14: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent and positive uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the product

$$\xi = \xi_1 \times \xi_2 \times \cdots \times \xi_n \tag{2.51}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \times \Phi_2^{-1}(\alpha) \times \cdots \times \Phi_n^{-1}(\alpha). \tag{2.52}$$

Exercise 2.15: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the minimum

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{2.53}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha) \wedge \cdots \wedge \Phi_n^{-1}(\alpha). \tag{2.54}$$

Exercise 2.16: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the maximum

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{2.55}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \vee \Phi_2^{-1}(\alpha) \vee \cdots \vee \Phi_n^{-1}(\alpha). \tag{2.56}$$

Example 2.12: The independence condition in Theorem 2.8 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then $\xi_1(\gamma) = \gamma$ is a linear uncertain variable with inverse uncertainty distribution

$$\Phi_1^{-1}(\alpha) = \alpha, \tag{2.57}$$

and $\xi_2(\gamma) = 1 - \gamma$ is also a linear uncertain variable with inverse uncertainty distribution

$$\Phi_2^{-1}(\alpha) = \alpha. \tag{2.58}$$

Note that ξ_1 and ξ_2 are not independent, and $\xi_1 + \xi_2 \equiv 1$ whose inverse uncertainty distribution is $\Psi^{-1}(\alpha) \equiv 1$. Thus

$$\Psi^{-1}(\alpha) \neq \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha). \tag{2.59}$$

Therefore, the independence condition cannot be removed.

Theorem 2.9 *Assume that* ξ_1 *and* ξ_2 *are independent linear uncertain variables* $\mathcal{L}(a_1, b_1)$ *and* $\mathcal{L}(a_2, b_2)$*, respectively. Then the sum* $\xi_1 + \xi_2$ *is also a linear uncertain variable* $\mathcal{L}(a_1 + a_2, b_1 + b_2)$*, i.e.,*

$$\mathcal{L}(a_1, b_1) + \mathcal{L}(a_2, b_2) = \mathcal{L}(a_1 + a_2, b_1 + b_2). \tag{2.60}$$

The product of a linear uncertain variable $\mathcal{L}(a, b)$ *and a scalar number* $k > 0$ *is also a linear uncertain variable* $\mathcal{L}(ka, kb)$*, i.e.,*

$$k \cdot \mathcal{L}(a, b) = \mathcal{L}(ka, kb). \tag{2.61}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = (1-\alpha)a_1 + \alpha b_1,$$

$$\Phi_2^{-1}(\alpha) = (1-\alpha)a_2 + \alpha b_2.$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (1-\alpha)(a_1 + a_2) + \alpha(b_1 + b_2).$$

Hence the sum is also a linear uncertain variable $\mathcal{L}(a_1 + a_2, b_1 + b_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{L}(a, b)$ is Φ. It follows from the operational law that when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = (1-\alpha)(ka) + \alpha(kb).$$

Hence $k\xi$ is just a linear uncertain variable $\mathcal{L}(ka, kb)$.

Theorem 2.10 *Assume that* ξ_1 *and* ξ_2 *are independent zigzag uncertain variables* $\mathcal{Z}(a_1, b_1, c_1)$ *and* $\mathcal{Z}(a_2, b_2, c_2)$*, respectively. Then the sum* $\xi_1 + \xi_2$ *is also a zigzag uncertain variable* $\mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$*, i.e.,*

$$\mathcal{Z}(a_1, b_1, c_1) + \mathcal{Z}(a_2, b_2, c_2) = \mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2). \tag{2.62}$$

The product of a zigzag uncertain variable $\mathcal{Z}(a, b, c)$ *and a scalar number* $k > 0$ *is also a zigzag uncertain variable* $\mathcal{Z}(ka, kb, kc)$*, i.e.,*

$$k \cdot \mathcal{Z}(a, b, c) = \mathcal{Z}(ka, kb, kc). \tag{2.63}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = \begin{cases} (1-2\alpha)a_1 + 2\alpha b_1, & \text{if } \alpha < 0.5 \\ (2-2\alpha)b_1 + (2\alpha - 1)c_1, & \text{if } \alpha \ge 0.5, \end{cases}$$

$$\Phi_2^{-1}(\alpha) = \begin{cases} (1-2\alpha)a_2 + 2\alpha b_2, & \text{if } \alpha < 0.5 \\ (2-2\alpha)b_2 + (2\alpha-1)c_2, & \text{if } \alpha \geq 0.5. \end{cases}$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \begin{cases} (1-2\alpha)(a_1+a_2) + 2\alpha(b_1+b_2), & \text{if } \alpha < 0.5 \\ (2-2\alpha)(b_1+b_2) + (2\alpha-1)(c_1+c_2), & \text{if } \alpha \geq 0.5. \end{cases}$$

Hence the sum is also a zigzag uncertain variable $\mathcal{Z}(a_1+a_2, b_1+b_2, c_1+c_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{Z}(a,b,c)$ is Φ. It follows from the operational law that when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)(ka) + 2\alpha(kb), & \text{if } \alpha < 0.5 \\ (2-2\alpha)(kb) + (2\alpha-1)(kc), & \text{if } \alpha \geq 0.5. \end{cases}$$

Hence $k\xi$ is just a zigzag uncertain variable $\mathcal{Z}(ka, kb, kc)$.

Theorem 2.11 *Let ξ_1 and ξ_2 be independent normal uncertain variables $\mathcal{N}(e_1, \sigma_1)$ and $\mathcal{N}(e_2, \sigma_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a normal uncertain variable $\mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2)$, i.e.,*

$$\mathcal{N}(e_1, \sigma_1) + \mathcal{N}(e_2, \sigma_2) = \mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2). \tag{2.64}$$

The product of a normal uncertain variable $\mathcal{N}(e, \sigma)$ and a scalar number $k > 0$ is also a normal uncertain variable $\mathcal{N}(ke, k\sigma)$, i.e.,

$$k \cdot \mathcal{N}(e, \sigma) = \mathcal{N}(ke, k\sigma). \tag{2.65}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = e_1 + \frac{\sigma_1\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha},$$

$$\Phi_2^{-1}(\alpha) = e_2 + \frac{\sigma_2\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (e_1 + e_2) + \frac{(\sigma_1+\sigma_2)\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Hence the sum is also a normal uncertain variable $\mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ is Φ. It follows from the operational law that, when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = (ke) + \frac{(k\sigma)\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Hence $k\xi$ is just a normal uncertain variable $\mathcal{N}(ke, k\sigma)$.

Theorem 2.12 *Assume that ξ_1 and ξ_2 are independent lognormal uncertain variables $\mathcal{LOGN}(e_1, \sigma_1)$ and $\mathcal{LOGN}(e_2, \sigma_2)$, respectively. Then the product $\xi_1 \cdot \xi_2$ is also a lognormal uncertain variable $\mathcal{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2)$, i.e.,*

$$\mathcal{LOGN}(e_1, \sigma_1) \cdot \mathcal{LOGN}(e_2, \sigma_2) = \mathcal{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2). \tag{2.66}$$

The product of a lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ and a scalar number $k > 0$ is also a lognormal uncertain variable $\mathcal{LOGN}(e + \ln k, \sigma)$, i.e.,

$$k \cdot \mathcal{LOGN}(e, \sigma) = \mathcal{LOGN}(e + \ln k, \sigma). \tag{2.67}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = \exp\left(e_1 + \frac{\sigma_1\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right),$$

$$\Phi_2^{-1}(\alpha) = \exp\left(e_2 + \frac{\sigma_2\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 \cdot \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \cdot \Phi_2^{-1}(\alpha) = \exp\left((e_1 + e_2) + \frac{(\sigma_1 + \sigma_2)\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

Hence the product is a lognormal uncertain variable $\mathcal{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{LOGN}(e, \sigma)$ is Φ. It follows from the operational law that, when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = \exp\left((e + \ln k) + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

Hence $k\xi$ is just a lognormal uncertain variable $\mathcal{LOGN}(e + \ln k, \sigma)$.

Remark 2.4: Keep in mind that the sum of lognormal uncertain variables is no longer lognormal.

Strictly Decreasing Function of Uncertain Variables

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly decreasing if

$$f(x_1, x_2, \cdots, x_n) \geq f(y_1, y_2, \cdots, y_n) \tag{2.68}$$

whenever $x_i \leq y_i$ for $i = 1, 2, \cdots, n$, and

$$f(x_1, x_2, \cdots, x_n) > f(y_1, y_2, \cdots, y_n) \tag{2.69}$$

whenever $x_i < y_i$ for $i = 1, 2, \cdots, n$. If $f(x_1, x_2, \cdots, x_n)$ is a strictly increasing function, then $-f(x_1, x_2, \cdots, x_n)$ is a strictly decreasing function. Furthermore, $1/f(x_1, x_2, \cdots, x_n)$ is also a strictly decreasing function provided that f is positive. Especially, the following are strictly decreasing functions,

$$f(x) = -x,$$
$$f(x) = \exp(-x),$$
$$f(x) = \frac{1}{x}, \quad x > 0.$$

Theorem 2.13 *(Liu [83]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly decreasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.70}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \tag{2.71}$$

Proof: For simplicity, we only prove the case $n = 2$. At first, we always have

$$\{\xi \leq \Psi^{-1}(\alpha)\} \equiv \{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha))\}.$$

On the one hand, since f is a strictly decreasing function, we obtain

$$\{\xi \leq \Psi^{-1}(\alpha)\} \supset \{\xi_1 \geq \Phi_1^{-1}(1-\alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\begin{aligned}\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} &\geq \mathcal{M}\{\xi_1 \geq \Phi_1^{-1}(1-\alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\} \\ &= \mathcal{M}\{\xi_1 \geq \Phi_1^{-1}(1-\alpha)\} \wedge \mathcal{M}\{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\} \\ &= \alpha \wedge \alpha = \alpha.\end{aligned}$$

On the other hand, since f is a strictly decreasing function, we obtain

$$\{\xi \leq \Psi^{-1}(\alpha)\} \subset \{\xi_1 \geq \Phi_1^{-1}(1-\alpha)\} \cup \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\begin{aligned}\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} &\le \mathcal{M}\{\xi_1 \ge \Phi_1^{-1}(1-\alpha)\} \cup \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} \\ &= \mathcal{M}\{\xi_1 \ge \Phi_1^{-1}(1-\alpha)\} \vee \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} \\ &= \alpha \vee \alpha = \alpha.\end{aligned}$$

It follows that $\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \alpha$. That is, Ψ^{-1} is just the inverse uncertainty distribution of ξ. The theorem is proved.

Exercise 2.17: Let ξ be a positive uncertain variable with regular uncertainty distribution Φ. Show that the reciprocal $1/\xi$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \frac{1}{\Phi^{-1}(1-\alpha)}. \tag{2.72}$$

Exercise 2.18: Let ξ be an uncertain variable with regular uncertainty distribution Φ. Show that $\exp(-\xi)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \exp\left(-\Phi^{-1}(1-\alpha)\right). \tag{2.73}$$

Exercise 2.19: Show that the independence condition in Theorem 2.13 cannot be removed.

Strictly Monotone Function of Uncertain Variables

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly monotone if it is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, that is,

$$f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \le f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n) \tag{2.74}$$

whenever $x_i \le y_i$ for $i = 1, 2, \cdots, m$ and $x_i \ge y_i$ for $i = m+1, m+2, \cdots, n$, and

$$f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) < f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n) \tag{2.75}$$

whenever $x_i < y_i$ for $i = 1, 2, \cdots, m$ and $x_i > y_i$ for $i = m+1, m+2, \cdots, n$. The following are strictly monotone functions,

$$\begin{gathered} f(x_1, x_2) = x_1 - x_2, \\ f(x_1, x_2) = x_1/x_2, \quad x_1, x_2 > 0, \\ f(x_1, x_2) = x_1/(x_1 + x_2), \quad x_1, x_2 > 0. \end{gathered}$$

Note that both strictly increasing function and strictly decreasing function are special cases of strictly monotone function.

Theorem 2.14 *(Liu [83]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.76}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \tag{2.77}$$

Proof: We only prove the case of $m = 1$ and $n = 2$. At first, we always have

$$\{\xi \le \Psi^{-1}(\alpha)\} \equiv \{f(\xi_1, \xi_2) \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha))\}.$$

On the one hand, since the function $f(x_1, x_2)$ is strictly increasing with respect to x_1 and strictly decreasing with x_2, we obtain

$$\{\xi \le \Psi^{-1}(\alpha)\} \supset \{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\begin{aligned}
\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} &\ge \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} \\
&= \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \wedge \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} \\
&= \alpha \wedge \alpha = \alpha.
\end{aligned}$$

On the other hand, since the function $f(x_1, x_2)$ is strictly increasing with respect to x_1 and strictly decreasing with x_2, we obtain

$$\{\xi \le \Psi^{-1}(\alpha)\} \subset \{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\begin{aligned}
\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} &\le \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} \\
&= \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \vee \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} \\
&= \alpha \vee \alpha = \alpha.
\end{aligned}$$

It follows that $\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \alpha$. That is, Ψ^{-1} is just the inverse uncertainty distribution of ξ. The theorem is proved.

Exercise 2.20: Let ξ_1 and ξ_2 be independent uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Show that the inverse uncertainty distribution of the difference $\xi_1 - \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha). \tag{2.78}$$

Exercise 2.21: Let ξ_1 and ξ_2 be independent and positive uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Show that the inverse uncertainty distribution of the quotient ξ_1/ξ_2 is

$$\Psi^{-1}(\alpha)=\frac{\Phi_1^{-1}(\alpha)}{\Phi_2^{-1}(1-\alpha)}. \tag{2.79}$$

Exercise 2.22: Assume ξ_1 and ξ_2 are independent and positive uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Show that the inverse uncertainty distribution of $\xi_1/(\xi_1+\xi_2)$ is

$$\Psi^{-1}(\alpha)=\frac{\Phi_1^{-1}(\alpha)}{\Phi_1^{-1}(\alpha)+\Phi_2^{-1}(1-\alpha)}. \tag{2.80}$$

Exercise 2.23: Show that the independence condition in Theorem 2.14 cannot be removed.

A Useful Theorem

In many cases, it is required to calculate $\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)\le 0\}$. Perhaps the first idea is to find the uncertainty distribution $\Psi(x)$ of $f(\xi_1,\xi_2,\cdots,\xi_n)$, and then the uncertain measure is just $\Psi(0)$. However, for convenience, we may use the following theorem.

Theorem 2.15 *(Liu [82]) Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If $f(\xi_1,\xi_2,\cdots,\xi_n)$ is strictly increasing with respect to $\xi_1,\xi_2,\cdots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\cdots,\xi_n$, then*

$$\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)\le 0\} \tag{2.81}$$

is the root α of the equation

$$f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))=0. \tag{2.82}$$

Proof: It follows from Theorem 2.14 that $f(\xi_1,\xi_2,\cdots,\xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha)=f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)).$$

Since $\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)\le 0\}=\Psi(0)$, it is the solution α of the equation $\Psi^{-1}(\alpha)=0$. The theorem is proved.

Remark 2.5: Keep in mind that sometimes the equation (2.82) may not have a root. In this case, if

$$f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))<0 \tag{2.83}$$

for all α, then we set the root $\alpha = 1$; and if

$$f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) > 0 \tag{2.84}$$

for all α, then we set the root $\alpha = 0$.

Remark 2.6: Since $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, the function

$$f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))$$

is strictly increasing with respect to α. See Figure 2.12. Thus its root α may be estimated by the bisection method:

Step 1. Set $a = 0$, $b = 1$ and $c = (a+b)/2$.

Step 2. If $f(\Phi_1^{-1}(c), \cdots, \Phi_m^{-1}(c), \Phi_{m+1}^{-1}(1-c), \cdots, \Phi_n^{-1}(1-c)) \le 0$, then set $a = c$. Otherwise, set $b = c$.

Step 3. If $|b - a| > \varepsilon$ (a predetermined precision), then set $c = (b-a)/2$ and go to Step 2. Otherwise, output c as the root.

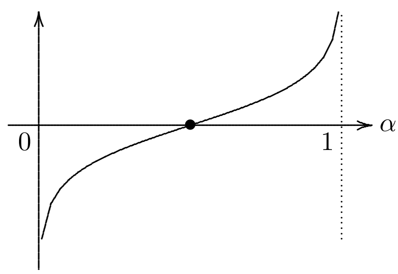


Figure 2.12: $f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))$

Exercise 2.24: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Assume the function $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$. Show that

$$\mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\} \tag{2.85}$$

is the root α of the equation

$$f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) = 0. \tag{2.86}$$

Exercise 2.25: Let ξ_1, ξ_2, ξ_3 be independent uncertain variables with regular uncertainty distributions Φ_1, Φ_2, Φ_3, respectively. Show that

$$\mathcal{M}\{\xi_1 \vee \xi_2 \ge \xi_3 + 5\} \tag{2.87}$$

is the root α of the equation

$$\Phi_1^{-1}(1-\alpha) \vee \Phi_2^{-1}(1-\alpha) = \Phi_3^{-1}(\alpha) + 5. \tag{2.88}$$

2.5 Operational Law: Distribution

This section will give some operational laws for calculating the uncertainty distributions of strictly increasing function, strictly decreasing function, and strictly monotone function of uncertain variables.

Strictly Increasing Function of Uncertain Variables

Theorem 2.16 *(Liu [83]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a continuous and strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.89}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \min_{1\le i\le n} \Phi_i(x_i). \tag{2.90}$$

Proof: For simplicity, we only prove the case $n = 2$. Since f is a continuous and strictly increasing function, it holds that

$$\{f(\xi_1, \xi_2) \le x\} = \bigcup_{f(x_1,x_2)=x} (\xi_1 \le x_1) \cap (\xi_2 \le x_2).$$

Thus the uncertainty distribution is

$$\Psi(x) = \mathcal{M}\{f(\xi_1, \xi_2) \le x\} = \mathcal{M}\left\{\bigcup_{f(x_1,x_2)=x} (\xi_1 \le x_1) \cap (\xi_2 \le x_2)\right\}.$$

Note that for each given number x, the event

$$\bigcup_{f(x_1,x_2)=x} (\xi_1 \le x_1) \cap (\xi_2 \le x_2)$$

is just a polyrectangle. It follows from the polyrectangular theorem that

$$\begin{aligned}\Psi(x) &= \sup_{f(x_1,x_2)=x} \mathcal{M}\{(\xi_1 \le x_1) \cap (\xi_2 \le x_2)\} \\ &= \sup_{f(x_1,x_2)=x} \mathcal{M}\{\xi_1 \le x_1\} \wedge \mathcal{M}\{\xi_2 \le x_2\} \\ &= \sup_{f(x_1,x_2)=x} \Phi_1(x_1) \wedge \Phi_2(x_2).\end{aligned}$$

The theorem is proved.

Remark 2.7: It is possible that the equation $f(x_1, x_2, \cdots, x_n) = x$ does not have a root for some values of x. In this case, if

$$f(x_1, x_2, \cdots, x_n) < x \tag{2.91}$$

for any vector $(x_1, x_2, \cdots, x_n)$, then we set $\Psi(x) = 1$; and if

$$f(x_1, x_2, \cdots, x_n) > x \tag{2.92}$$

for any vector $(x_1, x_2, \cdots, x_n)$, then we set $\Psi(x) = 0$.

Exercise 2.26: Let ξ be an uncertain variable with uncertainty distribution Φ, and let f be a continuous and strictly increasing function. Show that $f(\xi)$ has an uncertainty distribution

$$\Psi(x) = \Phi(f^{-1}(x)), \quad \forall x \in \Re. \tag{2.93}$$

Exercise 2.27: Let $\xi_1, \xi_2, \cdots, \xi_n$ be iid uncertain variables with a common uncertainty distribution Φ. Show that the sum

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \tag{2.94}$$

has an uncertainty distribution

$$\Psi(x) = \Phi\left(\frac{x}{n}\right). \tag{2.95}$$

Exercise 2.28: Let $\xi_1, \xi_2, \cdots, \xi_n$ be iid and positive uncertain variables with a common uncertainty distribution Φ. Show that the product

$$\xi = \xi_1 \xi_2 \cdots \xi_n \tag{2.96}$$

has an uncertainty distribution

$$\Psi(x) = \Phi\left(\sqrt[n]{x}\right). \tag{2.97}$$

Exercise 2.29: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the minimum

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{2.98}$$

has an uncertainty distribution

$$\Psi(x) = \Phi_1(x) \vee \Phi_2(x) \vee \cdots \vee \Phi_n(x). \tag{2.99}$$

Exercise 2.30: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the maximum

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{2.100}$$

has an uncertainty distribution

$$\Psi(x) = \Phi_1(x) \wedge \Phi_2(x) \wedge \cdots \wedge \Phi_n(x). \tag{2.101}$$

Example 2.13: The independence condition in Theorem 2.16 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Then $\xi_1(\gamma) = \gamma$ is a linear uncertain variable with uncertainty distribution

$$\Phi_1(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x > 1, \end{cases} \tag{2.102}$$

and $\xi_2(\gamma) = 1 - \gamma$ is also a linear uncertain variable with uncertainty distribution

$$\Phi_2(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x > 1. \end{cases} \tag{2.103}$$

Note that ξ_1 and ξ_2 are not independent, and $\xi_1 + \xi_2 \equiv 1$ whose uncertainty distribution is

$$\Psi(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1, & \text{if } x \ge 1. \end{cases} \tag{2.104}$$

Thus

$$\Psi(x) \ne \sup_{x_1 + x_2 = x} \Phi_1(x_1) \wedge \Phi_2(x_2). \tag{2.105}$$

Therefore, the independence condition cannot be removed.

Definition 2.15 *(Gao-Gao-Yang [47], Order Statistic) Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables, and let k be an index with $1 \le k \le n$. Then*

$$\xi = k\text{-min}[\xi_1, \xi_2, \cdots, \xi_n] \tag{2.106}$$

is called the kth order statistic of $\xi_1, \xi_2, \cdots, \xi_n$, where k-min represents the kth smallest value.

Theorem 2.17 *(Gao-Gao-Yang [47]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Then the kth order statistic of $\xi_1, \xi_2, \cdots, \xi_n$ has an uncertainty distribution*

$$\Psi(x) = k\text{-max}[\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)] \tag{2.107}$$

where k-max represents the kth largest value.

Proof: Since $f(x_1, x_2, \cdots, x_n) = k\text{-min}[x_1, x_2, \cdots, x_n]$ is a strictly increasing funtion, it follows from Theorem 2.16 that the kth order statistic has an uncertainty distribution

$$\begin{aligned}\Psi(x) &= \sup_{k\text{-min}[x_1,x_2,\cdots,x_n]=x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_n(x_n) \\ &= k\text{-max}[\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)].\end{aligned}$$

The theorem is proved.

Exercise 2.31: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Then

$$\xi = k\text{-max}[\xi_1, \xi_2, \cdots, \xi_n] \tag{2.108}$$

is just the $(n - k + 1)$th order statistic. Show that ξ has an uncertainty distribution

$$\Psi(x) = k\text{-min}[\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)]. \tag{2.109}$$

Theorem 2.18 *(Liu [89], Extreme Value Theorem) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables. Assume that*

$$S_i = \xi_1 + \xi_2 + \cdots + \xi_i \tag{2.110}$$

have uncertainty distributions Ψ_i for $i = 1, 2, \cdots, n$, respectively. Then the maximum

$$S = S_1 \vee S_2 \vee \cdots \vee S_n \tag{2.111}$$

has an uncertainty distribution

$$\Upsilon(x) = \Psi_1(x) \wedge \Psi_2(x) \wedge \cdots \wedge \Psi_n(x); \tag{2.112}$$

and the minimum

$$S = S_1 \wedge S_2 \wedge \cdots \wedge S_n \tag{2.113}$$

has an uncertainty distribution

$$\Upsilon(x) = \Psi_1(x) \vee \Psi_2(x) \vee \cdots \vee \Psi_n(x). \tag{2.114}$$

Proof: Assume that the uncertainty distributions of the uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. It follows from Theorem 2.16 that

$$\Psi_i(x) = \sup_{x_1+x_2+\cdots+x_i=x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_i(x_i)$$

for $i = 1, 2, \cdots, n$. Define

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee (x_1 + x_2) \vee \cdots \vee (x_1 + x_2 + \cdots + x_n).$$

Then f is a strictly increasing function and

$$S = f(\xi_1, \xi_2, \cdots, \xi_n).$$

It follows from Theorem 2.16 that S has an uncertainty distribution

$$\begin{aligned}\Upsilon(x) &= \sup_{f(x_1,x_2,\cdots,x_n)=x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_n(x_n)\\ &= \min_{1\le i\le n} \sup_{x_1+x_2+\cdots+x_i=x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_i(x_i)\\ &= \min_{1\le i\le n} \Psi_i(x).\end{aligned}$$

Thus (2.112) is verified. Similarly, define

$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge (x_1 + x_2) \wedge \cdots \wedge (x_1 + x_2 + \cdots + x_n).$$

Then f is a strictly increasing function and

$$S = f(\xi_1, \xi_2, \cdots, \xi_n).$$

It follows from Theorem 2.16 that S has an uncertainty distribution

$$\begin{aligned}\Upsilon(x) &= \sup_{f(x_1,x_2,\cdots,x_n)=x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_n(x_n)\\ &= \max_{1\le i\le n} \sup_{x_1+x_2+\cdots+x_i=x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_i(x_i)\\ &= \max_{1\le i\le n} \Psi_i(x).\end{aligned}$$

Thus (2.114) is verified.

Strictly Decreasing Function of Uncertain Variables

Theorem 2.19 *(Liu [83]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a continuous and strictly decreasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.115}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \min_{1\le i\le n} (1 - \Phi_i(x_i)). \tag{2.116}$$

Proof: For simplicity, we only prove the case $n = 2$. Since f is a continuous and strictly decreasing function, it holds that

$$\{f(\xi_1, \xi_2) \le x\} = \bigcup_{f(x_1,x_2)=x} (\xi_1 \ge x_1) \cap (\xi_2 \ge x_2).$$

Thus the uncertainty distribution is

$$\Psi(x) = \mathcal{M}\{f(\xi_1, \xi_2) \le x\} = \mathcal{M}\left\{ \bigcup_{f(x_1,x_2)=x} (\xi_1 \ge x_1) \cap (\xi_2 \ge x_2) \right\}.$$

Note that for each given number x, the event

$$\bigcup_{f(x_1,x_2)=x} (\xi_1 \ge x_1) \cap (\xi_2 \ge x_2)$$

is just a polyrectangle. It follows from the polyrectangular theorem that

$$\begin{aligned} \Psi(x) &= \sup_{f(x_1,x_2)=x} \mathcal{M}\left\{(\xi_1 \ge x_1) \cap (\xi_2 \ge x_2)\right\} \\ &= \sup_{f(x_1,x_2)=x} \mathcal{M}\{\xi_1 \ge x_1\} \wedge \mathcal{M}\{\xi_2 \ge x_2\} \\ &= \sup_{f(x_1,x_2)=x} (1 - \Phi_1(x_1)) \wedge (1 - \Phi_2(x_2)). \end{aligned}$$

The theorem is proved.

Exercise 2.32: Let ξ be an uncertain variable with continuous uncertainty distribution Φ, and let f be a continuous and strictly decreasing function. Show that $f(\xi)$ has an uncertainty distribution

$$\Psi(x) = 1 - \Phi(f^{-1}(x)), \quad \forall x \in \Re. \tag{2.117}$$

Exercise 2.33: Let ξ be an uncertain variable with continuous uncertainty distribution Φ, and let a and b be real numbers with $a < 0$. Show that $a\xi + b$ has an uncertainty distribution

$$\Psi(x) = 1 - \Phi\left(\frac{x-b}{a}\right), \quad \forall x \in \Re. \tag{2.118}$$

Exercise 2.34: Let ξ be a positive uncertain variable with continuous uncertainty distribution Φ. Show that $1/\xi$ has an uncertainty distribution

$$\Psi(x) = 1 - \Phi\left(\frac{1}{x}\right), \quad \forall x > 0. \tag{2.119}$$

Exercise 2.35: Let ξ be an uncertain variable with continuous uncertainty distribution Φ. Show that $\exp(-\xi)$ has an uncertainty distribution

$$\Psi(x) = 1 - \Phi(-\ln(x)), \quad \forall x > 0. \tag{2.120}$$

Exercise 2.36: Show that the independence condition in Theorem 2.19 cannot be removed.

Strictly Monotone Function of Uncertain Variables

Theorem 2.20 *(Liu [83]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is continuous, strictly increasing with respect to $\xi_1, \xi_2, \cdots,$ ξ_m and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.121}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \left(\min_{1 \le i \le m} \Phi_i(x_i) \wedge \min_{m+1 \le i \le n} (1 - \Phi_i(x_i)) \right). \tag{2.122}$$

Proof: For simplicity, we only prove the case of $m = 1$ and $n = 2$. Since $f(x_1, x_2)$ is continuous, strictly increasing with respect to x_1 and strictly decreasing with respect to x_2, it holds that

$$\{f(\xi_1, \xi_2) \le x\} = \bigcup_{f(x_1, x_2) = x} (\xi_1 \le x_1) \cap (\xi_2 \ge x_2).$$

Thus the uncertainty distribution is

$$\Psi(x) = \mathcal{M}\{f(\xi_1, \xi_2) \le x\} = \mathcal{M}\left\{ \bigcup_{f(x_1, x_2) = x} (\xi_1 \le x_1) \cap (\xi_2 \ge x_2) \right\}.$$

Note that for each given number x, the event

$$\bigcup_{f(x_1, x_2) = x} (\xi_1 \le x_1) \cap (\xi_2 \ge x_2)$$

is just a polyrectangle. It follows from the polyrectangular theorem that

$$\begin{aligned}
\Psi(x) &= \sup_{f(x_1, x_2) = x} \mathcal{M}\left\{(\xi_1 \le x_1) \cap (\xi_2 \ge x_2)\right\} \\
&= \sup_{f(x_1, x_2) = x} \mathcal{M}\{\xi_1 \le x_1\} \wedge \mathcal{M}\{\xi_2 \ge x_2\} \\
&= \sup_{f(x_1, x_2) = x} \Phi_1(x_1) \wedge (1 - \Phi_2(x_2)).
\end{aligned}$$

The theorem is proved.

Exercise 2.37: Let ξ_1 and ξ_2 be independent uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Show that $\xi_1 - \xi_2$ has an uncertainty distribution

$$\Psi(x) = \sup_{y \in \Re} \Phi_1(x + y) \wedge (1 - \Phi_2(y)). \tag{2.123}$$

Exercise 2.38: Let ξ_1 and ξ_2 be independent and positive uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Show that ξ_1/ξ_2 has an uncertainty distribution

$$\Psi(x) = \sup_{y>0} \Phi_1(xy) \wedge (1 - \Phi_2(y)). \tag{2.124}$$

Exercise 2.39: Let ξ_1 and ξ_2 be independent and positive uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Show that $\xi_1/(\xi_1 + \xi_2)$ has an uncertainty distribution

$$\Psi(x) = \sup_{y>0} \Phi_1(xy) \wedge (1 - \Phi_2(y - xy)). \tag{2.125}$$

Exercise 2.40: Show that the independence condition in Theorem 2.20 cannot be removed.

2.6 Operational Law: Boolean System

A function is said to be Boolean if it is a mapping from $\{0, 1\}^n$ to $\{0, 1\}$. For example,

$$f(x_1, x_2, x_3) = x_1 \vee x_2 \wedge x_3 \tag{2.126}$$

is a Boolean function. An uncertain variable is said to be Boolean if it takes values either 0 or 1. For example, the following is a Boolean uncertain variable,

$$\xi = \begin{cases} 1 \text{ with uncertain measure } a \\ 0 \text{ with uncertain measure } 1 - a \end{cases} \tag{2.127}$$

where a is a number between 0 and 1. This section introduces an operational law for Boolean system.

Theorem 2.21 *(Liu [83]) Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} 1 \textit{ with uncertain measure } a_i \\ 0 \textit{ with uncertain measure } 1 - a_i \end{cases} \tag{2.128}$$

for $i = 1, 2, \cdots, n$. If f is a Boolean function (not necessarily monotone), then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if } \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if } \displaystyle\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{2.129}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \tag{2.130}$$

for $i = 1, 2, \cdots, n$, respectively.

Proof: Let $B_1, B_2, \cdots, B_n$ be nonempty subsets of $\{0, 1\}$. In other words, they take values of $\{0\}$, $\{1\}$ or $\{0, 1\}$. Write

$$\Lambda = \{\xi = 1\}, \quad \Lambda^c = \{\xi = 0\}, \quad \Lambda_i = \{\xi_i \in B_i\}$$

for $i = 1, 2, \cdots, n$. It is easy to verify that

$$\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n = \Lambda \text{ if and only if } f(B_1, B_2, \cdots, B_n) = \{1\},$$

$$\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n = \Lambda^c \text{ if and only if } f(B_1, B_2, \cdots, B_n) = \{0\}.$$

It follows from the product axiom that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \sup\limits_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min\limits_{1 \le i \le n} \mathcal{M}\{\xi_i \in B_i\}, \\ \quad \text{if } \sup\limits_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min\limits_{1 \le i \le n} \mathcal{M}\{\xi_i \in B_i\} > 0.5 \\ 1 - \sup\limits_{f(B_1,B_2,\cdots,B_n)=\{0\}} \min\limits_{1 \le i \le n} \mathcal{M}\{\xi_i \in B_i\}, \\ \quad \text{if } \sup\limits_{f(B_1,B_2,\cdots,B_n)=\{0\}} \min\limits_{1 \le i \le n} \mathcal{M}\{\xi_i \in B_i\} > 0.5 \\ 0.5, \text{ otherwise.} \end{cases} \tag{2.131}$$

Please note that

$$\nu_i(1) = \mathcal{M}\{\xi_i = 1\}, \quad \nu_i(0) = \mathcal{M}\{\xi_i = 0\}$$

for $i = 1, 2, \cdots, n$. The argument breaks down into four cases. Case 1: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \le i \le n} \nu_i(x_i) < 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{0\}} \min_{1 \le i \le n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \le i \le n} \nu_i(x_i) > 0.5.$$

It follows from (2.131) that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \le i \le n} \nu_i(x_i).$$

Case 2: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \le i \le n} \nu_i(x_i) > 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

It follows from (2.131) that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i).$$

Case 3: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = 0.5,$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) = 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 0.5,$$

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{0\}} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 0.5.$$

It follows from (2.131) that

$$\mathcal{M}\{\xi = 1\} = 0.5 = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i).$$

Case 4: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = 0.5,$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) < 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

It follows from (2.131) that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i).$$

Hence the equation (2.129) is proved for the four cases.

Example 2.14: The independence condition in Theorem 2.21 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Then

$$\xi_1(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases} \tag{2.132}$$

is a Boolean uncertain variable with

$$\mathcal{M}\{\xi_1 = 1\} = 0.5, \tag{2.133}$$

and

$$\xi_2(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \end{cases} \tag{2.134}$$

is also a Boolean uncertain variable with

$$\mathcal{M}\{\xi_2 = 1\} = 0.5. \tag{2.135}$$

Note that ξ_1 and ξ_2 are not independent, and $\xi_1 \wedge \xi_2 \equiv 0$ from which we obtain

$$\mathcal{M}\{\xi_1 \wedge \xi_2 = 1\} = 0. \tag{2.136}$$

However, by using (2.129), we get

$$\mathcal{M}\{\xi_1 \wedge \xi_2 = 1\} = 0.5. \tag{2.137}$$

Thus the independence condition cannot be removed.

Theorem 2.22 *(Liu [83]), Order Statistic) Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} \text{1 with uncertain measure } a_i \\ \text{0 with uncertain measure } 1 - a_i \end{cases} \tag{2.138}$$

for $i = 1, 2, \cdots, n$. Then the kth order statistic

$$\xi = k\text{-min}\,[\xi_1, \xi_2, \cdots, \xi_n] \tag{2.139}$$

is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = k\text{-min}\,[a_1, a_2, \cdots, a_n]. \tag{2.140}$$

Proof: The corresponding Boolean function for the kth order statistic is

$$f(x_1, x_2, \cdots, x_n) = k\text{-min}\,[x_1, x_2, \cdots, x_n]. \tag{2.141}$$

Without loss of generality, we assume $a_1 \le a_2 \le \cdots \le a_n$. Then we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = a_k \wedge \min_{1\le i<k} (a_i \vee (1 - a_i)),$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) = (1 - a_k) \wedge \min_{k<i\le n} (a_i \vee (1 - a_i))$$

where $\nu_i(x_i)$ are defined by (2.130) for $i = 1, 2, \cdots, n$. When $a_k \ge 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) \ge 0.5,$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) = 1 - a_k.$$

It follows from Theorem 2.21 that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) = 1 - (1 - a_k) = a_k.$$

When $a_k < 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = a_k < 0.5.$$

It follows from Theorem 2.21 that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = a_k.$$

Thus $\mathcal{M}\{\xi = 1\}$ is always a_k, i.e., the kth smallest value of $a_1, a_2, \cdots, a_n$. The theorem is proved.

Exercise 2.41: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent Boolean uncertain variables defined by (2.138). Then the minimum

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{2.142}$$

is the 1st order statistic. Show that

$$\mathcal{M}\{\xi = 1\} = a_1 \wedge a_2 \wedge \cdots \wedge a_n. \tag{2.143}$$

Exercise 2.42: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent Boolean uncertain variables defined by (2.138). Then the maximum

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{2.144}$$

is the nth order statistic. Show that

$$\mathcal{M}\{\xi = 1\} = a_1 \vee a_2 \vee \cdots \vee a_n. \tag{2.145}$$

Exercise 2.43: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent Boolean uncertain variables defined by (2.138). Then

$$\xi = k\text{-max}\,[\xi_1, \xi_2, \cdots, \xi_n] \tag{2.146}$$

is the $(n - k + 1)$th order statistic. Show that

$$\mathcal{M}\{\xi = 1\} = k\text{-max}\,[a_1, a_2, \cdots, a_n]. \tag{2.147}$$

Boolean System Calculator

Boolean System Calculator is a function in the Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) for computing the uncertain measure like

$$\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)=1\} \tag{2.148}$$

where $\xi_1,\xi_2,\cdots,\xi_n$ are independent Boolean uncertain variables and f is a Boolean function. For example, let ξ_1,ξ_2,ξ_3 be independent Boolean uncertain variables,

$$\xi_1=\begin{cases}1 \text{ with uncertain mesure } 0.8\\ 0 \text{ with uncertain mesure } 0.2,\end{cases}$$

$$\xi_2=\begin{cases}1 \text{ with uncertain mesure } 0.7\\ 0 \text{ with uncertain mesure } 0.3,\end{cases}$$

$$\xi_3=\begin{cases}1 \text{ with uncertain mesure } 0.6\\ 0 \text{ with uncertain mesure } 0.4.\end{cases}$$

We also assume the Boolean function is

$$f(x_1,x_2,x_3)=\begin{cases}1, & \text{if } x_1+x_2+x_3=0 \text{ or } 2\\ 0, & \text{if } x_1+x_2+x_3=1 \text{ or } 3.\end{cases}$$

The Boolean System Calculator yields $\mathcal{M}\{f(\xi_1,\xi_2,\xi_3)=1\}=0.4$.

2.7 Expected Value

Expected value is the average value of uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable.

Definition 2.16 *(Liu [76]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi]=\int_0^{+\infty}\mathcal{M}\{\xi\ge x\}\mathrm{d}x-\int_{-\infty}^0\mathcal{M}\{\xi\le x\}\mathrm{d}x \tag{2.149}$$

provided that at least one of the two integrals is finite.

Theorem 2.23 *(Liu [76]) Let ξ be an uncertain variable with uncertainty distribution Φ. Then*

$$E[\xi]=\int_0^{+\infty}(1-\Phi(x))\mathrm{d}x-\int_{-\infty}^0\Phi(x)\mathrm{d}x. \tag{2.150}$$

Proof: It follows from the measure inversion theorem that for almost all numbers x, we have $\mathcal{M}\{\xi \ge x\} = 1 - \Phi(x)$ and $\mathcal{M}\{\xi \le x\} = \Phi(x)$. By using the definition of expected value operator, we obtain

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\}\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \end{aligned}$$

See Figure 2.13. The theorem is proved.

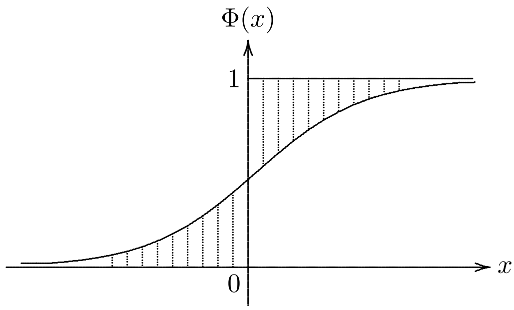


Figure 2.13: $E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x$

Theorem 2.24 *(Liu [83]) Let ξ be an uncertain variable with uncertainty distribution Φ. Then*

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \tag{2.151}$$

Proof: It follows from the integration by parts and Theorem 2.23 that the expected value is

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_0^{+\infty} x\mathrm{d}\Phi(x) + \int_{-\infty}^0 x\mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \end{aligned}$$

See Figure 2.14. The theorem is proved.

Remark 2.8: If the uncertainty distribution $\Phi(x)$ has a derivative $\phi(x)$, then we immediately have

$$E[\xi] = \int_{-\infty}^{+\infty} x\phi(x)\mathrm{d}x. \tag{2.152}$$

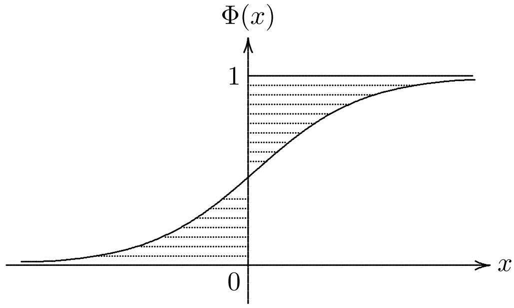


Figure 2.14: $E[\xi] = \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(x) = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha$

However, it is inappropriate to regard $\phi(x)$ as an uncertainty density function because uncertain measure is not additive, i.e., generally speaking,

$$\mathcal{M}\{a \le \xi \le b\} \ne \int_a^b \phi(x)\mathrm{d}x. \tag{2.153}$$

Theorem 2.25 *(Liu [83]) Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{2.154}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem 2.24 that the expected value is

$$E[\xi] = \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(x) = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

See Figure 2.14. The theorem is proved.

Exercise 2.44: Show that the linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ has an expected value

$$E[\xi] = \frac{a+b}{2}. \tag{2.155}$$

Exercise 2.45: Show that the zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ has an expected value

$$E[\xi] = \frac{a+2b+c}{4}. \tag{2.156}$$

Exercise 2.46: Show that the normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has an expected value e, i.e.,

$$E[\xi] = e. \tag{2.157}$$

Exercise 2.47: Show that the lognormal uncertain variable $\xi \sim \mathcal{LOGN}(e,\sigma)$ has an expected value

$$E[\xi]=\begin{cases}\sigma\sqrt{3}\exp(e)\csc(\sigma\sqrt{3}), & \text{if } \sigma<\pi/\sqrt{3}\\ +\infty, & \text{if } \sigma\ge\pi/\sqrt{3}.\end{cases} \tag{2.158}$$

This formula was first discovered by Dr. Zhongfeng Qin with the help of Maple software, and was verified again by Dr. Kai Yao through a rigorous mathematical derivation.

Exercise 2.48: Let ξ be an uncertain variable with empirical uncertainty distribution

$$\Phi(x)=\begin{cases}0, & \text{if } x<x_1\\ \alpha_i+\dfrac{(\alpha_{i+1}-\alpha_i)(x-x_i)}{x_{i+1}-x_i}, & \text{if } x_i\le x\le x_{i+1},\ 1\le i<n\\ 1, & \text{if } x>x_n\end{cases}$$

where $x_1<x_2<\cdots<x_n$ and $0\le\alpha_1\le\alpha_2\le\cdots\le\alpha_n\le 1$. Show that

$$E[\xi]=\frac{\alpha_1+\alpha_2}{2}x_1+\sum_{i=2}^{n-1}\frac{\alpha_{i+1}-\alpha_{i-1}}{2}x_i+\left(1-\frac{\alpha_{n-1}+\alpha_n}{2}\right)x_n. \tag{2.159}$$

Expected Value of Monotone Function of Uncertain Variables

Theorem 2.26 *(Liu-Ha [103]) Assume $\xi_1,\xi_2,\cdots,\xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If $f(\xi_1,\xi_2,\cdots,\xi_n)$ is strictly increasing with respect to $\xi_1,\xi_2,\cdots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\cdots,\xi_n$, then*

$$\xi=f(\xi_1,\xi_2,\cdots,\xi_n) \tag{2.160}$$

has an expected value

$$E[\xi]=\int_0^1 f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \tag{2.161}$$

Proof: Since the function $f(x_1,x_2,\cdots,x_n)$ is strictly increasing with respect to $x_1,x_2,\cdots,x_m$ and strictly decreasing with respect to $x_{m+1},x_{m+2},\cdots,x_n$, it follows from Theorem 2.14 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha)=f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)).$$

By using Theorem 2.25, we obtain (2.161). The theorem is proved.

Exercise 2.49: Let ξ be an uncertain variable with regular uncertainty distribution Φ, and let $f(x)$ be a strictly monotone (increasing or decreasing) function. Show that

$$E[f(\xi)] = \int_0^1 f(\Phi^{-1}(\alpha))\mathrm{d}\alpha. \tag{2.162}$$

Exercise 2.50: Let ξ be an uncertain variable with uncertainty distribution Φ, and let $f(x)$ be a strictly monotone (increasing or decreasing) function. Show that

$$E[f(\xi)] = \int_{-\infty}^{+\infty} f(x)\mathrm{d}\Phi(x). \tag{2.163}$$

Exercise 2.51: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$E[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha)\mathrm{d}\alpha. \tag{2.164}$$

Exercise 2.52: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$E\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)}\mathrm{d}\alpha. \tag{2.165}$$

Exercise 2.53: Assume ξ and η are independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$E\left[\frac{\xi}{\xi+\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha)+\Psi^{-1}(1-\alpha)}\mathrm{d}\alpha. \tag{2.166}$$

Linearity of Expected Value Operator

Theorem 2.27 *(Liu [83]) Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{2.167}$$

Proof: Without loss of generality, suppose ξ and η have regular uncertainty distributions Φ and Ψ, respectively. Otherwise, we may give the uncertainty distributions a small perturbation such that they become regular.

STEP 1: We first prove $E[a\xi] = aE[\xi]$. If $a = 0$, then the equation holds trivially. If $a > 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(\alpha).$$

It follows from Theorem 2.25 that

$$E[a\xi] = \int_0^1 a\Phi^{-1}(\alpha)\mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = aE[\xi].$$

If $a < 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(1-\alpha).$$

It follows from Theorem 2.25 that

$$E[a\xi] = \int_0^1 a\Phi^{-1}(1-\alpha)\mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = aE[\xi].$$

Thus we always have $E[a\xi] = aE[\xi]$.

STEP 2: We prove $E[\xi + \eta] = E[\xi] + E[\eta]$. The inverse uncertainty distribution of the sum $\xi + \eta$ is

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha).$$

It follows from Theorem 2.25 that

$$E[\xi+\eta] = \int_0^1 \Upsilon^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha = E[\xi] + E[\eta].$$

STEP 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$E[a\xi + b\eta] = E[a\xi] + E[b\eta] = aE[\xi] + bE[\eta].$$

The theorem is proved.

Example 2.15: Generally speaking, the expected value operator is not necessarily linear if the independence is not assumed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$ and $\mathcal{M}\{\gamma_3\} = 0.2$. Define two uncertain variables as follows,

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \\ 2, & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 2, & \text{if } \gamma = \gamma_2 \\ 3, & \text{if } \gamma = \gamma_3. \end{cases}$$

Note that ξ and η are not independent, and their sum is

$$(\xi+\eta)(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 2, & \text{if } \gamma = \gamma_2 \\ 5, & \text{if } \gamma = \gamma_3. \end{cases}$$

It is easy to verify that $E[\xi] = 0.9$, $E[\eta] = 1$ and $E[\xi + \eta] = 2$. Thus we have

$$E[\xi+\eta] > E[\xi] + E[\eta].$$

If the uncertain variables are defined by

$$\xi(\gamma)=\begin{cases}0, & \text{if } \gamma=\gamma_1\\ 1, & \text{if } \gamma=\gamma_2\\ 2, & \text{if } \gamma=\gamma_3,\end{cases} \qquad \eta(\gamma)=\begin{cases}0, & \text{if } \gamma=\gamma_1\\ 3, & \text{if } \gamma=\gamma_2\\ 1, & \text{if } \gamma=\gamma_3.\end{cases}$$

Then

$$(\xi+\eta)(\gamma)=\begin{cases}0, & \text{if } \gamma=\gamma_1\\ 4, & \text{if } \gamma=\gamma_2\\ 3, & \text{if } \gamma=\gamma_3.\end{cases}$$

It is easy to verify that $E[\xi]=0.6$, $E[\eta]=1$ and $E[\xi+\eta]=1.5$. Thus we have

$$E[\xi+\eta]<E[\xi]+E[\eta].$$

Therefore, the independence condition cannot be removed.

Comonotonic Functions of Uncertain Variable

Two real-valued functions f and g are said to be *comonotonic* if for any numbers x and y, we always have

$$(f(x)-f(y))(g(x)-g(y))\geq 0. \tag{2.168}$$

It is easy to verify that (i) any function is comonotonic with any positive constant multiple of the function; (ii) any monotone increasing functions are comonotonic with each other; and (iii) any monotone decreasing functions are also comonotonic with each other.

Theorem 2.28 *(Yang [164]) Let f and g be comonotonic functions. Then for any uncertain variable ξ, we have*

$$E[f(\xi)+g(\xi)]=E[f(\xi)]+E[g(\xi)]. \tag{2.169}$$

Proof: Without loss of generality, suppose $f(\xi)$ and $g(\xi)$ have regular uncertainty distributions Φ and Ψ, respectively. Otherwise, we may give the uncertainty distributions a small perturbation such that they become regular. Since f and g are comonotonic functions, at least one of the following relations is true,

$$\{f(\xi)\leq\Phi^{-1}(\alpha)\}\subset\{g(\xi)\leq\Psi^{-1}(\alpha)\},$$
$$\{f(\xi)\leq\Phi^{-1}(\alpha)\}\supset\{g(\xi)\leq\Psi^{-1}(\alpha)\}.$$

On the one hand, we have

$$\begin{aligned}&\mathcal{M}\{f(\xi)+g(\xi)\leq\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha)\}\\ \geq\ &\mathcal{M}\{(f(\xi)\leq\Phi^{-1}(\alpha))\cap(g(\xi)\leq\Psi^{-1}(\alpha))\}\\ =\ &\mathcal{M}\{f(\xi)\leq\Phi^{-1}(\alpha)\}\wedge\mathcal{M}\{g(\xi)\leq\Psi^{-1}(\alpha)\}\\ =\ &\alpha\wedge\alpha=\alpha.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathcal{M}\{f(\xi)+g(\xi)\le \Phi^{-1}(\alpha)+\Psi^{-1}(\alpha)\}\\
\le\ &\mathcal{M}\{(f(\xi)\le \Phi^{-1}(\alpha))\cup(g(\xi)\le \Psi^{-1}(\alpha))\}\\
=\ &\mathcal{M}\{f(\xi)\le \Phi^{-1}(\alpha)\}\vee\mathcal{M}\{g(\xi)\le \Psi^{-1}(\alpha)\}\\
=\ &\alpha\vee\alpha=\alpha.
\end{aligned}$$

It follows that

$$\mathcal{M}\{f(\xi)+g(\xi)\le \Phi^{-1}(\alpha)+\Psi^{-1}(\alpha)\}=\alpha$$

holds for each α. That is, $\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha)$ is the inverse uncertainty distribution of $f(\xi)+g(\xi)$. By using Theorem 2.25, we obtain

$$\begin{aligned}
E[f(\xi)+g(\xi)] &= \int_0^1 (\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha))\mathrm{d}\alpha\\
&= \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha+\int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha\\
&= E[f(\xi)]+E[g(\xi)].
\end{aligned}$$

The theorem is verified.

Exercise 2.54: Let ξ be a positive uncertain variable. Show that $\ln x$ and $\exp(x)$ are comonotonic functions on $(0,+\infty)$, and

$$E[\ln\xi+\exp(\xi)]=E[\ln\xi]+E[\exp(\xi)]. \tag{2.170}$$

Exercise 2.55: Let ξ be a positive uncertain variable. Show that $x, x^2, \cdots, x^n$ are comonotonic functions on $[0,+\infty)$, and

$$E[\xi+\xi^2+\cdots+\xi^n]=E[\xi]+E[\xi^2]+\cdots+E[\xi^n]. \tag{2.171}$$

Some Inequalities

Theorem 2.29 *(Liu [76]) Let ξ be an uncertain variable, and let f be a nonnegative even function. If f is decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$, then for any given number $t>0$, we have*

$$\mathcal{M}\{|\xi|\ge t\}\le \frac{E[f(\xi)]}{f(t)}. \tag{2.172}$$

Proof: It is clear that $\mathcal{M}\{|\xi| \geq f^{-1}(r)\}$ is a monotone decreasing function of r on $[0, \infty)$. It follows from the nonnegativity of $f(\xi)$ that

$$\begin{aligned}
E[f(\xi)] &= \int_0^{+\infty} \mathcal{M}\{f(\xi) \geq x\}\mathrm{d}x = \int_0^{+\infty} \mathcal{M}\{|\xi| \geq f^{-1}(x)\}\mathrm{d}x \\
&\geq \int_0^{f(t)} \mathcal{M}\{|\xi| \geq f^{-1}(x)\}\mathrm{d}x \geq \int_0^{f(t)} \mathcal{M}\{|\xi| \geq f^{-1}(f(t))\}\mathrm{d}x \\
&= \int_0^{f(t)} \mathcal{M}\{|\xi| \geq t\}\mathrm{d}x = f(t) \cdot \mathcal{M}\{|\xi| \geq t\}
\end{aligned}$$

which proves the inequality.

Theorem 2.30 *(Liu [76], Markov Inequality) Let ξ be an uncertain variable. Then for any given numbers $t > 0$ and $p > 0$, we have*

$$\mathcal{M}\{|\xi| \geq t\} \leq \frac{E[|\xi|^p]}{t^p}. \tag{2.173}$$

Proof: It is a special case of Theorem 2.29 when $f(x) = |x|^p$.

Example 2.16: For any given positive number t, we define an uncertain variable as follows,

$$\xi = \begin{cases} 0 \text{ with uncertain measure } 1/2 \\ t \text{ with uncertain measure } 1/2. \end{cases}$$

Then $E[\xi^p] = t^p/2$ and $\mathcal{M}\{\xi \geq t\} = 1/2 = E[\xi^p]/t^p$.

Theorem 2.31 *(Liu [76], Hölder's Inequality) Let p and q be positive numbers with $1/p + 1/q = 1$, and let ξ and η be independent uncertain variables. Then*

$$E[|\xi\eta|] \leq \sqrt[p]{E[|\xi|^p]}\sqrt[q]{E[|\eta|^q]}. \tag{2.174}$$

Proof: The inequality holds trivially if at least one of ξ and η is zero a.s. Now we assume $E[|\xi|^p] > 0$ and $E[|\eta|^q] > 0$. It is easy to prove that the function $f(x, y) = \sqrt[p]{x}\sqrt[q]{y}$ is a concave function on $\{(x, y) : x \geq 0, y \geq 0\}$. Thus for any point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$, there exist two real numbers a and b such that

$$f(x, y) - f(x_0, y_0) \leq a(x - x_0) + b(y - y_0), \quad \forall x \geq 0, y \geq 0.$$

Letting $x_0 = E[|\xi|^p]$, $y_0 = E[|\eta|^q]$, $x = |\xi|^p$ and $y = |\eta|^q$, we have

$$f(|\xi|^p, |\eta|^q) - f(E[|\xi|^p], E[|\eta|^q]) \leq a(|\xi|^p - E[|\xi|^p]) + b(|\eta|^q - E[|\eta|^q]).$$

Taking the expected values on both sides, we obtain

$$E[f(|\xi|^p, |\eta|^q)] \leq f(E[|\xi|^p], E[|\eta|^q]).$$

Hence the inequality (2.174) holds.

Theorem 2.32 *(Liu [76], Minkowski Inequality) Let p be a real number with $p \ge 1$, and let ξ and η be independent uncertain variables. Then*

$$\sqrt[p]{E[|\xi+\eta|^p]} \le \sqrt[p]{E[|\xi|^p]} + \sqrt[p]{E[|\eta|^p]}. \tag{2.175}$$

Proof: The inequality holds trivially if at least one of ξ and η is zero a.s. Now we assume $E[|\xi|^p] > 0$ and $E[|\eta|^p] > 0$. It is easy to prove that the function $f(x, y) = (\sqrt[p]{x} + \sqrt[p]{y})^p$ is a concave function on $\{(x, y) : x \ge 0, y \ge 0\}$. Thus for any point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$, there exist two real numbers a and b such that

$$f(x, y) - f(x_0, y_0) \le a(x - x_0) + b(y - y_0), \quad \forall x \ge 0, y \ge 0.$$

Letting $x_0 = E[|\xi|^p]$, $y_0 = E[|\eta|^p]$, $x = |\xi|^p$ and $y = |\eta|^p$, we have

$$f(|\xi|^p, |\eta|^p) - f(E[|\xi|^p], E[|\eta|^p]) \le a(|\xi|^p - E[|\xi|^p]) + b(|\eta|^p - E[|\eta|^p]).$$

Taking the expected values on both sides, we obtain

$$E[f(|\xi|^p, |\eta|^p)] \le f(E[|\xi|^p], E[|\eta|^p]).$$

Hence the inequality (2.175) holds.

Theorem 2.33 *(Liu [76], Jensen's Inequality) Let ξ be an uncertain variable, and let f be a convex function. Then*

$$f(E[\xi]) \le E[f(\xi)]. \tag{2.176}$$

Especially, when $f(x) = |x|^p$ and $p \ge 1$, we have $|E[\xi]|^p \le E[|\xi|^p]$.

Proof: Since f is a convex function, for each y, there exists a number k such that $f(x) - f(y) \ge k \cdot (x - y)$. Replacing x with ξ and y with $E[\xi]$, we obtain

$$f(\xi) - f(E[\xi]) \ge k \cdot (\xi - E[\xi]).$$

Taking the expected values on both sides, we have

$$E[f(\xi)] - f(E[\xi]) \ge k \cdot (E[\xi] - E[\xi]) = 0$$

which proves the inequality.

Exercise 2.56: (Zhang [199]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with finite expected values, and let f be a convex function. Show that

$$f(E[\xi_1], E[\xi_2], \cdots, E[\xi_n]) \le E[f(\xi_1, \xi_2, \cdots, \xi_n)]. \tag{2.177}$$

2.8 Variance

The variance of uncertain variable provides a degree of the spread of the distribution around its expected value. A small value of variance indicates that the uncertain variable is tightly concentrated around its expected value; and a large value of variance indicates that the uncertain variable has a wide spread around its expected value.

Definition 2.17 *(Liu [76]) Let ξ be an uncertain variable with finite expected value e. Then the variance of ξ is*

$$V[\xi] = E[(\xi - e)^2]. \tag{2.178}$$

This definition tells us that the variance is just the expected value of $(\xi - e)^2$. Since $(\xi - e)^2$ is a nonnegative uncertain variable, we also have

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\}\mathrm{d}x. \tag{2.179}$$

Theorem 2.34 *(Liu [76]) If ξ is an uncertain variable with finite expected value, a and b are real numbers, then*

$$V[a\xi + b] = a^2 V[\xi]. \tag{2.180}$$

Proof: Let e be the expected value of ξ. Then $a\xi + b$ has an expected value $ae + b$. It follows from the definition of variance that

$$V[a\xi + b] = E\left[(a\xi + b - (ae + b))^2\right] = a^2 E[(\xi - e)^2] = a^2 V[\xi].$$

The theorem is thus verified.

Theorem 2.35 *(Liu [76]) Let ξ be an uncertain variable with expected value e. Then $V[\xi] = 0$ if and only if $\mathcal{M}\{\xi = e\} = 1$. That is, the uncertain variable ξ is essentially the constant e.*

Proof: We first assume $V[\xi] = 0$. It follows from the equation (2.179) that

$$\int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\}\mathrm{d}x = 0$$

which implies $\mathcal{M}\{(\xi - e)^2 \geq x\} = 0$ for any $x > 0$. Hence we have

$$\mathcal{M}\{(\xi - e)^2 = 0\} = 1.$$

That is, $\mathcal{M}\{\xi = e\} = 1$. Conversely, assume $\mathcal{M}\{\xi = e\} = 1$. Then we immediately have $\mathcal{M}\{(\xi - e)^2 = 0\} = 1$ and $\mathcal{M}\{(\xi - e)^2 \geq x\} = 0$ for any $x > 0$. Thus

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\}\mathrm{d}x = 0.$$

The theorem is proved.

Theorem 2.36 *(Yao [175]) Let ξ and η be independent uncertain variables whose variances exist. Then*

$$\sqrt{V[\xi+\eta]} \le \sqrt{V[\xi]} + \sqrt{V[\eta]}. \tag{2.181}$$

Proof: It is a special case of Theorem 2.32 when $p = 2$ and the uncertain variables ξ and η are replaced with $\xi - E[\xi]$ and $\eta - E[\eta]$, respectively.

Theorem 2.37 *(Liu [76], Chebyshev Inequality) Let ξ be an uncertain variable whose variance exists. Then for any given number $t > 0$, we have*

$$\mathcal{M}\{|\xi - E[\xi]| \ge t\} \le \frac{V[\xi]}{t^2}. \tag{2.182}$$

Proof: It is a special case of Theorem 2.29 when the uncertain variable ξ is replaced with $\xi - E[\xi]$, and $f(x) = x^2$.

Example 2.17: For any given positive number t, we define an uncertain variable as follows,

$$\xi = \begin{cases} -t \text{ with uncertain measure } 1/2 \\ t \text{ with uncertain measure } 1/2. \end{cases}$$

Then $V[\xi] = t^2$ and $\mathcal{M}\{|\xi - E[\xi]| \ge t\} = 1 = V[\xi]/t^2$.

How to Obtain Variance from Uncertainty Distribution?

Let ξ be an uncertain variable with expected value e. If we only know its uncertainty distribution Φ, then the variance

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \ge x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathcal{M}\{(\xi \ge e + \sqrt{x}) \cup (\xi \le e - \sqrt{x})\}\mathrm{d}x \\ &\le \int_0^{+\infty} (\mathcal{M}\{\xi \ge e + \sqrt{x}\} + \mathcal{M}\{\xi \le e - \sqrt{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x. \end{aligned}$$

Thus we have the following stipulation.

Stipulation 2.1 *(Liu [83]) Let ξ be an uncertain variable with uncertainty distribution Φ and finite expected value e. Then*

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x. \tag{2.183}$$

Theorem 2.38 *(Liu [94]) Let ξ be an uncertain variable with uncertainty distribution Φ and finite expected value e. Then*

$$V[\xi]=\int_{-\infty}^{+\infty}(x-e)^2\mathrm{d}\Phi(x). \tag{2.184}$$

Proof: This theorem is based on Stipulation 2.1 that says the variance of ξ is

$$V[\xi]=\int_0^{+\infty}(1-\Phi(e+\sqrt{y}))\mathrm{d}y+\int_0^{+\infty}\Phi(e-\sqrt{y})\mathrm{d}y.$$

Substituting $e+\sqrt{y}$ with x and y with $(x-e)^2$, the change of variables and integration by parts produce

$$\int_0^{+\infty}(1-\Phi(e+\sqrt{y}))\mathrm{d}y=\int_e^{+\infty}(1-\Phi(x))\mathrm{d}(x-e)^2=\int_e^{+\infty}(x-e)^2\mathrm{d}\Phi(x).$$

Similarly, substituting $e-\sqrt{y}$ with x and y with $(x-e)^2$, we obtain

$$\int_0^{+\infty}\Phi(e-\sqrt{y})\mathrm{d}y=\int_e^{-\infty}\Phi(x)\mathrm{d}(x-e)^2=\int_{-\infty}^{e}(x-e)^2\mathrm{d}\Phi(x).$$

It follows that the variance is

$$V[\xi]=\int_e^{+\infty}(x-e)^2\mathrm{d}\Phi(x)+\int_{-\infty}^{e}(x-e)^2\mathrm{d}\Phi(x)=\int_{-\infty}^{+\infty}(x-e)^2\mathrm{d}\Phi(x).$$

The theorem is verified.

Theorem 2.39 *(Yao [175]) Let ξ be an uncertain variable with regular uncertainty distribution Φ and finite expected value e. Then*

$$V[\xi]=\int_0^1(\Phi^{-1}(\alpha)-e)^2\mathrm{d}\alpha. \tag{2.185}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem 2.38 that the variance is

$$V[\xi]=\int_{-\infty}^{+\infty}(x-e)^2\mathrm{d}\Phi(x)=\int_0^1(\Phi^{-1}(\alpha)-e)^2\mathrm{d}\alpha.$$

The theorem is verified.

Exercise 2.57: Show that the linear uncertain variable $\xi\sim\mathcal{L}(a,b)$ has a variance

$$V[\xi]=\frac{(b-a)^2}{12}. \tag{2.186}$$

Exercise 2.58: Show that the normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has a variance

$$V[\xi] = \sigma^2. \tag{2.187}$$

Exercise 2.59: Let ξ and η be independent uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Assume there exist two real numbers a and b such that

$$\Phi^{-1}(\alpha) = a\Psi^{-1}(\alpha) + b \tag{2.188}$$

for all $\alpha \in (0, 1)$. Show that

$$\sqrt{V[\xi + \eta]} = \sqrt{V[\xi]} + \sqrt{V[\eta]} \tag{2.189}$$

in the sense of Stipulation 2.1.

Remark 2.9: If ξ and η are independent linear uncertain variables, then the condition (2.188) is met. If they are independent normal uncertain variables, then the condition (2.188) is also met.

2.9 Moment

Definition 2.18 *(Liu [76]) Let ξ be an uncertain variable and let k be a positive integer. Then $E[\xi^k]$ is called the k-th moment of ξ.*

Theorem 2.40 *(Liu [94]) Let ξ be an uncertain variable with uncertainty distribution Φ, and let k be an odd number. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}))\mathrm{d}x - \int_{-\infty}^0 \Phi(\sqrt[k]{x})\mathrm{d}x. \tag{2.190}$$

Proof: Since k is an odd number, it follows from the definition of expected value operator that

$$\begin{aligned}
E[\xi^k] &= \int_0^{+\infty} \mathcal{M}\{\xi^k \geq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi^k \leq x\}\mathrm{d}x \\
&= \int_0^{+\infty} \mathcal{M}\{\xi \geq \sqrt[k]{x}\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \leq \sqrt[k]{x}\}\mathrm{d}x \\
&= \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}))\mathrm{d}x - \int_{-\infty}^0 \Phi(\sqrt[k]{x})\mathrm{d}x.
\end{aligned}$$

The theorem is proved.

However, when k is an even number, the k-th moment of ξ cannot be uniquely determined by the uncertainty distribution Φ. In this case, we have

$$\begin{aligned} E[\xi^k] &= \int_0^{+\infty} \mathcal{M}\{\xi^k \ge x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathcal{M}\{(\xi \ge \sqrt[k]{x}) \cup (\xi \le -\sqrt[k]{x})\}\mathrm{d}x \\ &\le \int_0^{+\infty} (\mathcal{M}\{\xi \ge \sqrt[k]{x}\} + \mathcal{M}\{\xi \le -\sqrt[k]{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}) + \Phi(-\sqrt[k]{x}))\mathrm{d}x. \end{aligned}$$

Thus for the even number k, we have the following stipulation.

Stipulation 2.2 *(Liu [94]) Let ξ be an uncertain variable with uncertainty distribution Φ, and let k be an even number. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}) + \Phi(-\sqrt[k]{x}))\mathrm{d}x. \tag{2.191}$$

Theorem 2.41 *(Liu [94]) Let ξ be an uncertain variable with uncertainty distribution Φ, and let k be a positive integer. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x). \tag{2.192}$$

Proof: When k is an odd number, Theorem 2.40 says that the k-th moment is

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))\mathrm{d}y - \int_{-\infty}^0 \Phi(\sqrt[k]{y})\mathrm{d}y.$$

Substituting $\sqrt[k]{y}$ with x and y with x^k, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))\mathrm{d}y = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x^k = \int_0^{+\infty} x^k \mathrm{d}\Phi(x)$$

and

$$\int_{-\infty}^0 \Phi(\sqrt[k]{y})\mathrm{d}y = \int_{-\infty}^0 \Phi(x)\mathrm{d}x^k = -\int_{-\infty}^0 x^k \mathrm{d}\Phi(x).$$

Thus we have

$$E[\xi^k] = \int_0^{+\infty} x^k \mathrm{d}\Phi(x) + \int_{-\infty}^0 x^k \mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x).$$

When k is an even number, the theorem is based on Stipulation 2.2 that says the k-th moment is

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}) + \Phi(-\sqrt[k]{y}))\mathrm{d}y.$$

Substituting $\sqrt[k]{y}$ with x and y with x^k, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))\mathrm{d}y = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x^k = \int_0^{+\infty} x^k \mathrm{d}\Phi(x).$$

Similarly, substituting $-\sqrt[k]{y}$ with x and y with x^k, we obtain

$$\int_0^{+\infty} \Phi(-\sqrt[k]{y})\mathrm{d}y = \int_{-\infty}^{0} \Phi(x)\mathrm{d}x^k = \int_{-\infty}^{0} x^k \mathrm{d}\Phi(x).$$

It follows that the k-th moment is

$$E[\xi^k] = \int_0^{+\infty} x^k \mathrm{d}\Phi(x) + \int_{-\infty}^{0} x^k \mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x).$$

The theorem is thus verified for any positive integer k.

Theorem 2.42 *(Sheng-Kar [138]) Let ξ be an uncertain variable with regular uncertainty distribution Φ, and let k be a positive integer. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_0^1 (\Phi^{-1}(\alpha))^k \mathrm{d}\alpha. \tag{2.193}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem 2.41 that the k-th moment is

$$E[\xi^k] = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha))^k \mathrm{d}\alpha.$$

The theorem is verified.

Exercise 2.60: Show that the second moment of linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ is

$$E[\xi^2] = \frac{a^2 + ab + b^2}{3}. \tag{2.194}$$

Exercise 2.61: Show that the second moment of normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ is

$$E[\xi^2] = e^2 + \sigma^2. \tag{2.195}$$

2.10 Distance

Definition 2.19 *(Liu [76]) The distance between uncertain variables ξ and η is defined as*

$$d(\xi,\eta)=E[|\xi-\eta|]. \tag{2.196}$$

That is, the distance between ξ and η is just the expected value of $|\xi-\eta|$. Since $|\xi-\eta|$ is a nonnegative uncertain variable, we always have

$$d(\xi,\eta)=\int_0^{+\infty}\mathcal{M}\{|\xi-\eta|\ge x\}\mathrm{d}x. \tag{2.197}$$

Theorem 2.43 *(Liu [76]) Let ξ,η,τ be uncertain variables, and let $d(\cdot,\cdot)$ be the distance. Then we have*
(a) (Nonnegativity) $d(\xi,\eta)\ge 0$;
(b) (Identification) $d(\xi,\eta)=0$ if and only if $\xi=\eta$;
(c) (Symmetry) $d(\xi,\eta)=d(\eta,\xi)$;
(d) (Triangle Inequality) $d(\xi,\eta)\le 2d(\xi,\tau)+2d(\eta,\tau)$.

Proof: The parts (a), (b) and (c) follow immediately from the definition. Now we prove the part (d). It follows from the subadditivity axiom that

$$\begin{aligned}
d(\xi,\eta) &= \int_0^{+\infty}\mathcal{M}\left\{|\xi-\eta|\ge x\right\}\mathrm{d}x\\
&\le \int_0^{+\infty}\mathcal{M}\left\{|\xi-\tau|+|\tau-\eta|\ge x\right\}\mathrm{d}x\\
&\le \int_0^{+\infty}\mathcal{M}\left\{(|\xi-\tau|\ge x/2)\cup(|\tau-\eta|\ge x/2)\right\}\mathrm{d}x\\
&\le \int_0^{+\infty}\left(\mathcal{M}\{|\xi-\tau|\ge x/2\}+\mathcal{M}\{|\tau-\eta|\ge x/2\}\right)\mathrm{d}x\\
&= 2E[|\xi-\tau|]+2E[|\tau-\eta|]=2d(\xi,\tau)+2d(\tau,\eta).
\end{aligned}$$

Example 2.18: Let $\Gamma=\{\gamma_1,\gamma_2,\gamma_3\}$. Define $\mathcal{M}\{\emptyset\}=0$, $\mathcal{M}\{\Gamma\}=1$ and $\mathcal{M}\{\Lambda\}=1/2$ for any subset Λ (excluding $\emptyset$ and Γ). We set uncertain variables ξ, η and τ as follows,

$$\xi(\gamma)=\begin{cases}1, & \text{if } \gamma=\gamma_1\\ 1, & \text{if } \gamma=\gamma_2\\ 0, & \text{if } \gamma=\gamma_3,\end{cases}\qquad \eta(\gamma)=\begin{cases}0, & \text{if } \gamma=\gamma_1\\ -1, & \text{if } \gamma=\gamma_2\\ -1, & \text{if } \gamma=\gamma_3,\end{cases}\qquad \tau(\gamma)\equiv 0.$$

It is easy to verify that $d(\xi,\tau)=d(\tau,\eta)=0.5$ and $d(\xi,\eta)=1.5$. Thus

$$d(\xi,\eta)=1.5(d(\xi,\tau)+d(\tau,\eta)).$$

A conjecture is $d(\xi,\eta)\le 1.5(d(\xi,\tau)+d(\tau,\eta))$ for arbitrary uncertain variables ξ, η and τ. This is an open problem.

How to Obtain Distance from Uncertainty Distributions?

Let ξ and η be independent uncertain variables. If $\xi-\eta$ has an uncertainty distribution Υ, then the distance between ξ and η is

$$\begin{aligned} d(\xi,\eta) &= \int_0^{+\infty} \mathcal{M}\{|\xi-\eta|\ge x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathcal{M}\{(\xi-\eta\ge x)\cup(\xi-\eta\le -x)\}\mathrm{d}x \\ &\le \int_0^{+\infty} (\mathcal{M}\{\xi-\eta\ge x\}+\mathcal{M}\{\xi-\eta\le -x\})\mathrm{d}x \\ &= \int_0^{+\infty} (1-\Upsilon(x)+\Upsilon(-x))\mathrm{d}x. \end{aligned}$$

Thus we have the following stipulation.

Stipulation 2.3 *(Liu [94]) Let ξ and η be independent uncertain variables, and let Υ be the uncertainty distribution of $\xi-\eta$. Then the distance between ξ and η is*

$$d(\xi,\eta) = \int_0^{+\infty} (1-\Upsilon(x)+\Upsilon(-x))\mathrm{d}x. \tag{2.198}$$

Theorem 2.44 *(Liu [94]) Let ξ and η be independent uncertain variables, and let Υ be the uncertainty distribution of $\xi-\eta$. Then the distance between ξ and η is*

$$d(\xi,\eta) = \int_{-\infty}^{+\infty} |x|\mathrm{d}\Upsilon(x). \tag{2.199}$$

Proof: This theorem is based on Stipulation 2.3. The change of variables and integration by parts produce

$$\begin{aligned} d(\xi,\eta) &= \int_0^{+\infty} (1-\Upsilon(x)+\Upsilon(-x))\mathrm{d}x \\ &= \int_0^{+\infty} x\mathrm{d}\Upsilon(x) - \int_0^{+\infty} x\mathrm{d}\Upsilon(-x) \\ &= \int_0^{+\infty} |x|\mathrm{d}\Upsilon(x) + \int_{-\infty}^{0} |x|\mathrm{d}\Upsilon(x) \\ &= \int_{-\infty}^{+\infty} |x|\mathrm{d}\Upsilon(x). \end{aligned}$$

The theorem is proved.

Exercise 2.62: Let ξ be an uncertain variable with uncertainty distribution Φ, and let c be a constant. Show that the distance between ξ and c is

$$d(\xi,c) = \int_{-\infty}^{+\infty} |x-c|\mathrm{d}\Phi(x). \tag{2.200}$$

Theorem 2.45 *(Liu [94]) Let ξ and η be independent uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Then the distance between ξ and η is*

$$d(\xi,\eta) = \int_0^1 |\Upsilon^{-1}(\alpha)|\mathrm{d}\alpha. \tag{2.201}$$

where $\Upsilon^{-1}(\alpha)$ is the inverse uncertainty distribution of $\xi - \eta$, and

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) - \Psi^{-1}(1-\alpha). \tag{2.202}$$

Proof: Substituting $\Upsilon(x)$ with α and x with $\Upsilon^{-1}(\alpha)$, it follows from the change of variables and Theorem 2.44 that the distance is

$$d(\xi,\eta) = \int_{-\infty}^{+\infty} |x|\mathrm{d}\Upsilon(x) = \int_0^1 |\Upsilon^{-1}(\alpha)|\mathrm{d}\alpha.$$

The theorem is verified.

Exercise 2.63: Let ξ be an uncertain variable with regular uncertainty distribution Φ, and let c be a constant. Show that the distance between ξ and c is

$$d(\xi,c) = \int_0^1 |\Phi^{-1}(\alpha) - c|\mathrm{d}\alpha. \tag{2.203}$$

2.11 Entropy

This section provides a definition of entropy to characterize the uncertainty of uncertain variables.

Definition 2.20 *(Liu [79]) Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its entropy is defined by*

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x \tag{2.204}$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

Example 2.19: Let ξ be an uncertain variable with uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{if } x \ge a. \end{cases} \tag{2.205}$$

Essentially, ξ is a constant a. It follows from the definition of entropy that

$$H[\xi] = -\int_{-\infty}^{a} (0\ln 0 + 1\ln 1)\,\mathrm{d}x - \int_a^{+\infty} (1\ln 1 + 0\ln 0)\,\mathrm{d}x = 0.$$

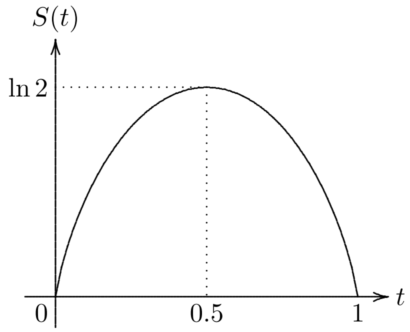


Figure 2.15: Function $S(t) = -t \ln t - (1-t)\ln(1-t)$. It is easy to verify that $S(t)$ is a symmetric function about $t = 0.5$, strictly increasing on the interval $[0, 0.5]$, strictly decreasing on the interval $[0.5, 1]$, and reaches its unique maximum $\ln 2$ at $t = 0.5$.

This means a constant has no uncertainty.

Example 2.20: Let ξ be a linear uncertain variable $\mathcal{L}(a, b)$. Then its entropy is

$$H[\xi] = -\int_a^b \left(\frac{x-a}{b-a} \ln \frac{x-a}{b-a} + \frac{b-x}{b-a} \ln \frac{b-x}{b-a} \right) \mathrm{d}x = \frac{b-a}{2}. \tag{2.206}$$

Exercise 2.64: Show that the zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ has an entropy

$$H[\xi] = \frac{c-a}{2}. \tag{2.207}$$

Exercise 2.65: Show that the normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has an entropy

$$H[\xi] = \frac{\pi\sigma}{\sqrt{3}}. \tag{2.208}$$

Theorem 2.46 *Let ξ be an uncertain variable. Then $H[\xi] \ge 0$ and equality holds if ξ is essentially a constant.*

Proof: The nonnegativity is clear. In addition, when an uncertain variable tends to a constant, its entropy tends to the minimum 0.

Theorem 2.47 *Let ξ be an uncertain variable taking values on the interval $[a, b]$. Then*

$$H[\xi] \le (b-a)\ln 2 \tag{2.209}$$

and equality holds if ξ has an uncertainty distribution $\Phi(x) = 0.5$ on $[a, b]$.

Proof: The theorem follows from the fact that the function $S(t)$ reaches its maximum $\ln 2$ at $t = 0.5$.

Theorem 2.48 *Let ξ be an uncertain variable, and let c be a real number. Then*

$$H[\xi + c] = H[\xi]. \tag{2.210}$$

That is, the entropy is invariant under arbitrary translations.

Proof: Write the uncertainty distribution of ξ by Φ. Then the uncertain variable $\xi + c$ has an uncertainty distribution $\Phi(x - c)$. It follows from the definition of entropy that

$$H[\xi + c] = \int_{-\infty}^{+\infty} S\left(\Phi(x - c)\right) \mathrm{d}x = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x = H[\xi].$$

The theorem is proved.

Theorem 2.49 *(Dai-Chen [19]) Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then*

$$H[\xi] = \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha} \mathrm{d}\alpha. \tag{2.211}$$

Proof: It is clear that $S(\alpha)$ is a derivable function whose derivative has the form

$$S'(\alpha) = -\ln \frac{\alpha}{1 - \alpha}.$$

Since

$$S(\Phi(x)) = \int_0^{\Phi(x)} S'(\alpha)\mathrm{d}\alpha = -\int_{\Phi(x)}^1 S'(\alpha)\mathrm{d}\alpha,$$

we have

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x = \int_{-\infty}^0 \int_0^{\Phi(x)} S'(\alpha)\mathrm{d}\alpha\mathrm{d}x - \int_0^{+\infty} \int_{\Phi(x)}^1 S'(\alpha)\mathrm{d}\alpha\mathrm{d}x.$$

It follows from Fubini theorem that

$$\begin{aligned}
H[\xi] &= \int_0^{\Phi(0)} \int_{\Phi^{-1}(\alpha)}^0 S'(\alpha)\mathrm{d}x\mathrm{d}\alpha - \int_{\Phi(0)}^1 \int_0^{\Phi^{-1}(\alpha)} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha \\
&= -\int_0^{\Phi(0)} \Phi^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha - \int_{\Phi(0)}^1 \Phi^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha \\
&= -\int_0^1 \Phi^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1 - \alpha}\mathrm{d}\alpha.
\end{aligned}$$

The theorem is verified.

Theorem 2.50 *(Dai-Chen [19]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.212}$$

has an entropy

$$H[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha.$$

Proof: Since $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots,$ x_m and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, it follows from Theorem 2.14 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

By using Theorem 2.49, we get the entropy formula.

Exercise 2.66: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$H[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha.$$

Exercise 2.67: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$H\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)} \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha.$$

Exercise 2.68: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$H\left[\frac{\xi}{\xi+\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi^{-1}(1-\alpha)} \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha.$$

Theorem 2.51 *(Dai-Chen [19]) Let ξ and η be independent uncertain variables. Then for any real numbers a and b, we have*

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta]. \tag{2.213}$$

Proof: Without loss of generality, suppose ξ and η have regular uncertainty distributions Φ and Ψ, respectively. Otherwise, we may give the uncertainty distributions a small perturbation such that they become regular.

STEP 1: We prove $H[a\xi]=|a|H[\xi]$. If $a>0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha)=a\Phi^{-1}(\alpha).$$

It follows from Theorem 2.49 that

$$H[a\xi]=\int_0^1 a\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=a\int_0^1\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=|a|H[\xi].$$

If $a=0$, then we immediately have $H[a\xi]=0=|a|H[\xi]$. If $a<0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha)=a\Phi^{-1}(1-\alpha).$$

It follows from Theorem 2.49 that

$$H[a\xi]=\int_0^1 a\Phi^{-1}(1-\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=(-a)\int_0^1\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=|a|H[\xi].$$

Thus we always have $H[a\xi]=|a|H[\xi]$.

STEP 2: We prove $H[\xi+\eta]=H[\xi]+H[\eta]$. Note that the inverse uncertainty distribution of $\xi+\eta$ is

$$\Upsilon^{-1}(\alpha)=\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha).$$

It follows from Theorem 2.49 that

$$H[\xi+\eta]=\int_0^1(\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha))\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=H[\xi]+H[\eta].$$

STEP 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$H[a\xi+b\eta]=H[a\xi]+H[b\eta]=|a|H[\xi]+|b|H[\eta].$$

The theorem is proved.

Example 2.21: The independence condition in Theorem 2.51 cannot be removed. For example, take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Then $\xi(\gamma)=\gamma$ is a linear uncertain variable $\mathcal{L}(0,1)$ with entropy

$$H[\xi]=0.5, \tag{2.214}$$

and $\eta(\gamma)=1-\gamma$ is also a linear uncertain variable $\mathcal{L}(0,1)$ with entropy

$$H[\eta]=0.5. \tag{2.215}$$

Note that ξ and η are not independent, and $\xi+\eta\equiv 1$ whose entropy is

$$H[\xi+\eta]=0. \tag{2.216}$$

Thus

$$H[\xi+\eta]\neq H[\xi]+H[\eta]. \tag{2.217}$$

Therefore, the independence condition cannot be removed.

Maximum Entropy Principle

Given some constraints, for example, expected value and variance, there are usually multiple compatible uncertainty distributions. Which uncertainty distribution shall we take? The *maximum entropy principle* attempts to select the uncertainty distribution that has maximum entropy and satisfies the prescribed constraints.

Theorem 2.52 *(Chen-Dai [8]) Let ξ be an uncertain variable whose uncertainty distribution is arbitrary but the expected value e and variance σ^2. Then*

$$H[\xi] \le \frac{\pi\sigma}{\sqrt{3}} \tag{2.218}$$

and the equality holds if ξ is a normal uncertain variable $\mathcal{N}(e, \sigma)$.

Proof: Let $\Phi(x)$ be the uncertainty distribution of ξ and write $\Psi(x) = \Phi(2e - x)$ for $x \ge e$. It follows from the stipulation (2.1) and the change of variable of integral that the variance is

$$V[\xi] = 2\int_e^{+\infty} (x-e)(1-\Phi(x))\mathrm{d}x + 2\int_e^{+\infty} (x-e)\Psi(x)\mathrm{d}x = \sigma^2.$$

Thus there exists a real number κ such that

$$2\int_e^{+\infty} (x-e)(1-\Phi(x))\mathrm{d}x = \kappa\sigma^2,$$

$$2\int_e^{+\infty} (x-e)\Psi(x)\mathrm{d}x = (1-\kappa)\sigma^2.$$

The maximum entropy distribution Φ should maximize the entropy

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x = \int_e^{+\infty} S(\Phi(x))\mathrm{d}x + \int_e^{+\infty} S(\Psi(x))\mathrm{d}x$$

subject to the above two constraints. The Lagrangian is

$$\begin{aligned} L = &\int_e^{+\infty} S(\Phi(x))\mathrm{d}x + \int_e^{+\infty} S(\Psi(x))\mathrm{d}x \\ &-\alpha\left(2\int_e^{+\infty} (x-e)(1-\Phi(x))\mathrm{d}x - \kappa\sigma^2\right) \\ &-\beta\left(2\int_e^{+\infty} (x-e)\Psi(x)\mathrm{d}x - (1-\kappa)\sigma^2\right). \end{aligned}$$

The maximum entropy distribution meets Euler-Lagrange equations

$$\ln\Phi(x) - \ln(1-\Phi(x)) = 2\alpha(x-e),$$

$$\ln \Psi(x) - \ln(1 - \Psi(x)) = 2\beta(e - x).$$

Thus Φ and Ψ have the forms

$$\Phi(x) = (1 + \exp(2\alpha(e - x)))^{-1},$$
$$\Psi(x) = (1 + \exp(2\beta(x - e)))^{-1}.$$

Substituting them into the variance constraints, we get

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{6\kappa}\sigma}\right)\right)^{-1},$$

$$\Psi(x) = \left(1 + \exp\left(\frac{\pi(x - e)}{\sqrt{6(1-\kappa)}\sigma}\right)\right)^{-1}.$$

Then the entropy is

$$H[\xi] = \frac{\pi\sigma\sqrt{\kappa}}{\sqrt{6}} + \frac{\pi\sigma\sqrt{1-\kappa}}{\sqrt{6}}$$

which achieves the maximum when $\kappa = 1/2$. Thus the maximum entropy distribution is just the normal uncertainty distribution $\mathcal{N}(e, \sigma)$.

2.12 Conditional Uncertainty Distribution

Definition 2.21 *(Liu [76]) The conditional uncertainty distribution Φ of an uncertain variable ξ given A is defined by*

$$\Phi(x|A) = \mathcal{M}\{\xi \le x|A\} \tag{2.219}$$

provided that $\mathcal{M}\{A\} > 0$.

Theorem 2.53 *(Liu [83]) Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$, and let t be a real number with $\Phi(t) < 1$. Then the conditional uncertainty distribution of ξ given $\xi > t$ is*

$$\Phi(x|(t, +\infty)) = \begin{cases} 0, & \text{if } \Phi(x) \le \Phi(t) \\ \dfrac{\Phi(x)}{1 - \Phi(t)} \wedge 0.5, & \text{if } \Phi(t) < \Phi(x) \le (1 + \Phi(t))/2 \\ \dfrac{\Phi(x) - \Phi(t)}{1 - \Phi(t)}, & \text{if } (1 + \Phi(t))/2 \le \Phi(x). \end{cases}$$

Proof: It follows from $\Phi(x|(t, +\infty)) = \mathcal{M}\{\xi \le x|\xi > t\}$ and the definition of conditional uncertainty that

$$\Phi(x|(t,+\infty)) = \begin{cases} \dfrac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

When $\Phi(x) \le \Phi(t)$, we have $x \le t$, and

$$\frac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{\mathcal{M}\{\emptyset\}}{1 - \Phi(t)} = 0 < 0.5.$$

Thus

$$\Phi(x|(t, +\infty)) = \frac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = 0.$$

When $\Phi(t) < \Phi(x) \le (1 + \Phi(t))/2$, we have $x > t$, and

$$\frac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{1 - \Phi(x)}{1 - \Phi(t)} \ge \frac{1 - (1 + \Phi(t))/2}{1 - \Phi(t)} = 0.5$$

and

$$\frac{\mathcal{M}\{(\xi \le x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} \le \frac{\Phi(x)}{1 - \Phi(t)}.$$

It follows from the maximum uncertainty principle that

$$\Phi(x|(t, +\infty)) = \frac{\Phi(x)}{1 - \Phi(t)} \wedge 0.5.$$

When $(1 + \Phi(t))/2 \le \Phi(x)$, we have $x \ge t$, and

$$\frac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{1 - \Phi(x)}{1 - \Phi(t)} \le \frac{1 - (1 + \Phi(t))/2}{1 - \Phi(t)} \le 0.5.$$

Thus

$$\Phi(x|(t, +\infty)) = 1 - \frac{\mathcal{M}\{(\xi > x) \cap (\xi > t)\}}{\mathcal{M}\{\xi > t\}} = 1 - \frac{1 - \Phi(x)}{1 - \Phi(t)} = \frac{\Phi(x) - \Phi(t)}{1 - \Phi(t)}.$$

The theorem is proved.

Exercise 2.69: Let ξ be a linear uncertain variable $\mathcal{L}(a, b)$, and let t be a real number with $a < t < b$. Show that the conditional uncertainty distribution of ξ given $\xi > t$ is

$$\Phi(x|(t, +\infty)) = \begin{cases} 0, & \text{if } x \le t \\ \dfrac{x - a}{b - t} \wedge 0.5, & \text{if } t < x \le (b + t)/2 \\ \dfrac{x - t}{b - t} \wedge 1, & \text{if } (b + t)/2 \le x. \end{cases}$$

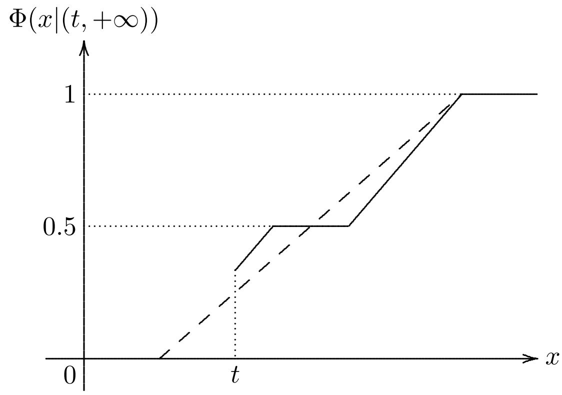


Figure 2.16: Conditional Uncertainty Distribution $\Phi(x|(t,+\infty))$

Theorem 2.54 *(Liu [83]) Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$, and let t be a real number with $\Phi(t)>0$. Then the conditional uncertainty distribution of ξ given $\xi\le t$ is*

$$\Phi(x|(-\infty,t])=\begin{cases}\dfrac{\Phi(x)}{\Phi(t)}, & \text{if } \Phi(x)\le\Phi(t)/2\\ \dfrac{\Phi(x)+\Phi(t)-1}{\Phi(t)}\vee 0.5, & \text{if } \Phi(t)/2\le\Phi(x)<\Phi(t)\\ 1, & \text{if } \Phi(t)\le\Phi(x).\end{cases}$$

Proof: It follows from $\Phi(x|(-\infty,t])=\mathcal{M}\{\xi\le x|\xi\le t\}$ and the definition of conditional uncertainty that

$$\Phi(x|(-\infty,t])=\begin{cases}\dfrac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}<0.5\\ 1-\dfrac{\mathcal{M}\{(\xi> x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi> x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}<0.5\\ 0.5, & \text{otherwise.}\end{cases}$$

When $\Phi(x)\le\Phi(t)/2$, we have $x<t$, and

$$\frac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}=\frac{\Phi(x)}{\Phi(t)}\le\frac{\Phi(t)/2}{\Phi(t)}=0.5.$$

Thus

$$\Phi(x|(-\infty,t])=\frac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}=\frac{\Phi(x)}{\Phi(t)}.$$

When $\Phi(t)/2\le\Phi(x)<\Phi(t)$, we have $x<t$, and

$$\frac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}=\frac{\Phi(x)}{\Phi(t)}\ge\frac{\Phi(t)/2}{\Phi(t)}=0.5$$

and

$$\frac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} \le \frac{1 - \Phi(x)}{\Phi(t)},$$

i.e.,

$$1 - \frac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} \ge \frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)}.$$

It follows from the maximum uncertainty principle that

$$\Phi(x|(-\infty, t]) = \frac{\Phi(x) + \Phi(t) - 1}{\Phi(t)} \vee 0.5.$$

When $\Phi(t) \le \Phi(x)$, we have $x \ge t$, and

$$\frac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} = \frac{\mathcal{M}\{\emptyset\}}{\Phi(t)} = 0 < 0.5.$$

Thus

$$\Phi(x|(-\infty, t]) = 1 - \frac{\mathcal{M}\{(\xi > x) \cap (\xi \le t)\}}{\mathcal{M}\{\xi \le t\}} = 1 - 0 = 1.$$

The theorem is proved.

Exercise 2.70: Let ξ be a linear uncertain variable $\mathcal{L}(a, b)$, and let t be a real number with $a < t < b$. Show that the conditional uncertainty distribution of ξ given $\xi \le t$ is

$$\Phi(x|(-\infty, t]) = \begin{cases} \dfrac{x-a}{t-a} \vee 0, & \text{if } x \le (a+t)/2 \\ \left(1 - \dfrac{b-x}{t-a}\right) \vee 0.5, & \text{if } (a+t)/2 \le x < t \\ 1, & \text{if } x \ge t. \end{cases}$$

2.13 Uncertain Sequence

Uncertain sequence is a sequence of uncertain variables indexed by integers. This section introduces four convergence concepts of uncertain sequence: convergence almost surely (a.s.), convergence in measure, convergence in mean, and convergence in distribution.

Definition 2.22 *(Liu [76]) The uncertain sequence $\{\xi_i\}$ is said to be convergent a.s. to ξ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{i \to \infty} |\xi_i(\gamma) - \xi(\gamma)| = 0 \tag{2.220}$$

for every $\gamma \in \Lambda$. In that case we write $\xi_i \to \xi$, a.s.

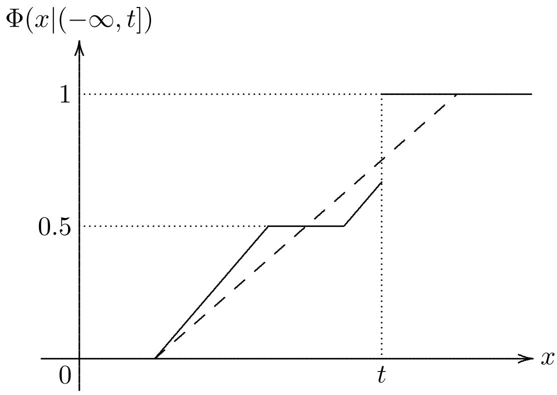


Figure 2.17: Conditional Uncertainty Distribution $\Phi(x|(-\infty,t])$

Table 2.1: Relationship among Convergence Concepts

Convergence in Mean	$\Rightarrow$	Convergence in Measure	$\Rightarrow$	Convergence in Distribution
		Convergence Almost Surely		

Definition 2.23 *(Liu [76]) The uncertain sequence $\{\xi_i\}$ is said to be convergent in measure to ξ if*

$$\lim_{i\to\infty} \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = 0 \tag{2.221}$$

for every $\varepsilon > 0$.

Definition 2.24 *(Liu [76]) The uncertain sequence $\{\xi_i\}$ is said to be convergent in mean to ξ if*

$$\lim_{i\to\infty} E[|\xi_i - \xi|] = 0. \tag{2.222}$$

Definition 2.25 *(Liu [76]) Let $\Phi, \Phi_1, \Phi_2, \cdots$ be the uncertainty distributions of uncertain variables $\xi, \xi_1, \xi_2, \cdots$, respectively. We say the uncertain sequence $\{\xi_i\}$ converges in distribution to ξ if*

$$\lim_{i\to\infty} \Phi_i(x) = \Phi(x) \tag{2.223}$$

for all x at which $\Phi(x)$ is continuous.

Convergence in Mean vs. Convergence in Measure

Theorem 2.55 *(Liu [76]) If the uncertain sequence $\{\xi_i\}$ converges in mean to ξ, then $\{\xi_i\}$ converges in measure to ξ.*

Proof: Since $\{\xi_i\}$ converges in mean to ξ, we have $E[|\xi_i-\xi|]\to 0$ as $i\to\infty$. For any given number $\varepsilon>0$, it follows from Markov inequality that

$$\mathcal{M}\{|\xi_i-\xi|\ge\varepsilon\}\le\frac{E[|\xi_i-\xi|]}{\varepsilon}\to 0$$

as $i\to\infty$. Thus $\{\xi_i\}$ converges in measure to ξ. The theorem is proved.

Example 2.22: Convergence in measure does not imply convergence in mean. Take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2,\cdots\}$ with power set and

$$\mathcal{M}\{\Lambda\}=\sum_{\gamma_j\in\Lambda}\frac{1}{2^j}.$$

Define uncertain variables as

$$\xi_i(\gamma_j)=\begin{cases}2^i, & \text{if } j=i\\ 0, & \text{otherwise}\end{cases}$$

for $i=1,2,\cdots$ and $\xi\equiv 0$. For any small number $\varepsilon>0$, we have

$$\mathcal{M}\{|\xi_i-\xi|\ge\varepsilon\}=\mathcal{M}\{|\xi_i-\xi|\ge\varepsilon\}=\frac{1}{2^i}\to 0$$

as $i\to\infty$. That is, the sequence $\{\xi_i\}$ converges in measure to ξ. However, for each i, we have

$$E[|\xi_i-\xi|]=1.$$

That is, the sequence $\{\xi_i\}$ does not converge in mean to ξ.

Convergence in Measure vs. Convergence in Distribution

Theorem 2.56 *(Liu [76]) If the uncertain sequence $\{\xi_i\}$ converges in measure to ξ, then $\{\xi_i\}$ converges in distribution to ξ.*

Proof: Let x be a continuity point of the uncertainty distribution Φ. On the one hand, for any $y>x$, we have

$$\{\xi_i\le x\}=\{\xi_i\le x,\xi\le y\}\cup\{\xi_i\le x,\xi>y\}\subset\{\xi\le y\}\cup\{|\xi_i-\xi|\ge y-x\}.$$

It follows from the subadditivity axiom that

$$\Phi_i(x)\le\Phi(y)+\mathcal{M}\{|\xi_i-\xi|\ge y-x\}.$$

Since $\{\xi_i\}$ converges in measure to ξ, we have $\mathcal{M}\{|\xi_i-\xi|\ge y-x\}\to 0$ as $i\to\infty$. Thus we obtain $\limsup_{i\to\infty}\Phi_i(x)\le\Phi(y)$ for any $y>x$. Letting $y\to x$, we get

$$\limsup_{i\to\infty}\Phi_i(x)\le\Phi(x). \tag{2.224}$$

On the other hand, for any $z < x$, we have

$$\{\xi \le z\} = \{\xi_i \le x, \xi \le z\} \cup \{\xi_i > x, \xi \le z\} \subset \{\xi_i \le x\} \cup \{|\xi_i - \xi| \ge x - z\}$$

which implies that

$$\Phi(z) \le \Phi_i(x) + \mathcal{M}\{|\xi_i - \xi| \ge x - z\}.$$

Since $\mathcal{M}\{|\xi_i - \xi| \ge x - z\} \to 0$, we obtain $\Phi(z) \le \liminf_{i\to\infty} \Phi_i(x)$ for any $z < x$. Letting $z \to x$, we get

$$\Phi(x) \le \liminf_{i\to\infty} \Phi_i(x). \tag{2.225}$$

It follows from (2.224) and (2.225) that $\Phi_i(x) \to \Phi(x)$ as $i \to \infty$. The theorem is proved.

Example 2.23: Convergence in distribution does not imply convergence in measure. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 1/2$. Define uncertain variables as

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2, \end{cases}$$

and $\xi_i = -\xi$ for $i = 1, 2, \cdots$ Then ξ_i and ξ have the same uncertainty distribution. Thus $\{\xi_i\}$ converges in distribution to ξ. However, for some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \ge \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \ge \varepsilon\} = 1.$$

That is, the sequence $\{\xi_i\}$ does not converge in measure to ξ.

Convergence Almost Surely vs. Convergence in Measure

Example 2.24: Convergence a.s. does not imply convergence in measure. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise.} \end{cases}$$

Define uncertain variables as

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then the sequence $\{\xi_i\}$ converges a.s. to ξ. However, for some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \ge \varepsilon\} = 0.5$$

for each i. That is, the sequence $\{\xi_i\}$ does not converge in measure to ξ.

Example 2.25: Convergence in measure does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. For any positive integer i, there is an integer j such that $i = 2^j + k$, where k is an integer between 0 and $2^j - 1$. Define uncertain variables as

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if } k/2^j \le \gamma \le (k+1)/2^j \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then for any small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \ge \varepsilon\} = \frac{1}{2^j} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in measure to ξ. However, for any $\gamma \in [0,1]$, there is an infinite number of intervals of the form $[k/2^j, (k+1)/2^j]$ containing γ. Thus $\xi_i(\gamma)$ does not converge to 0. In other words, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Convergence Almost Surely vs. Convergence in Mean

Example 2.26: Convergence a.s. does not imply convergence in mean. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_j \in \Lambda} \frac{1}{2^j}.$$

Define uncertain variables as

$$\xi_i(\gamma_j) = \begin{cases} 2^i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then ξ_i converges a.s. to ξ. However, the sequence $\{\xi_i\}$ does not converge in mean to ξ because $E[|\xi_i - \xi|] \equiv 1$ for each i.

Example 2.27: Convergence in mean does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. For any positive integer i, there is an integer j such that $i = 2^j + k$, where k is an integer between 0 and $2^j - 1$. Define uncertain variables as

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if } k/2^j \le \gamma \le (k+1)/2^j \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then

$$E[|\xi_i - \xi|] = \frac{1}{2^j} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in mean to ξ. However, for any $\gamma \in [0,1]$, there is an infinite number of intervals of the form $[k/2^j, (k+1)/2^j]$ containing γ. Thus $\xi_i(\gamma)$ does not converge to 0. In other words, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Convergence Almost Surely vs. Convergence in Distribution

Example 2.28: Convergence in distribution does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 1/2$. Define uncertain variables as

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases}$$

and $\xi_i = -\xi$ for $i = 1, 2, \cdots$ Then ξ_i and ξ have the same uncertainty distribution. Thus $\{\xi_i\}$ converges in distribution to ξ. However, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Example 2.29: Convergence a.s. does not imply convergence in distribution. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise.} \end{cases}$$

Define uncertain variables as

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then the sequence $\{\xi_i\}$ converges a.s. to ξ. However, the uncertainty distributions of ξ_i are

$$\Phi_i(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.5, & \text{if } 0 \le x < i \\ 1, & \text{if } x \ge i \end{cases}$$

for $i = 1, 2, \cdots$, respectively, and the uncertainty distribution of ξ is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \ge 0. \end{cases}$$

It is clear that $\Phi_i(x)$ does not converge to $\Phi(x)$ at $x > 0$. That is, the sequence $\{\xi_i\}$ does not converge in distribution to ξ.

2.14 Uncertain Vector

As an extension of uncertain variable, this section introduces a concept of uncertain vector whose components are uncertain variables.

Definition 2.26 *(Liu [76]) A k-dimensional uncertain vector is a function $\boldsymbol{\xi}$ from an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of k-dimensional real vectors such that $\{\boldsymbol{\xi}\in B\}$ is an event for any Borel set B of k-dimensional real vectors.*

Theorem 2.57 *(Liu [76]) The vector $(\xi_1,\xi_2,\cdots,\xi_k)$ is an uncertain vector if and only if $\xi_1,\xi_2,\cdots,\xi_k$ are uncertain variables.*

Proof: Write $\boldsymbol{\xi}=(\xi_1,\xi_2,\cdots,\xi_k)$. Suppose that $\boldsymbol{\xi}$ is an uncertain vector on the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$. For any Borel set B of real numbers, the set $B\times\Re^{k-1}$ is a Borel set of k-dimensional real vectors. Thus the set

$$\{\xi_1\in B\}=\{\xi_1\in B,\xi_2\in\Re,\cdots,\xi_k\in\Re\}=\{\boldsymbol{\xi}\in B\times\Re^{k-1}\}$$

is an event. Hence ξ_1 is an uncertain variable. A similar process may prove that $\xi_2,\xi_3,\cdots,\xi_k$ are uncertain variables.

Conversely, suppose that all $\xi_1,\xi_2,\cdots,\xi_k$ are uncertain variables on the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$. We define

$$\mathcal{B}=\left\{B\subset\Re^k \mid \{\boldsymbol{\xi}\in B\}\text{ is an event}\right\}.$$

The vector $\boldsymbol{\xi}=(\xi_1,\xi_2,\cdots,\xi_k)$ is proved to be an uncertain vector if we can prove that $\mathcal{B}$ contains all Borel sets of k-dimensional real vectors. First, the class $\mathcal{B}$ contains all open intervals of $\Re^k$ because

$$\left\{\boldsymbol{\xi}\in\prod_{i=1}^{k}(a_i,b_i)\right\}=\bigcap_{i=1}^{k}\{\xi_i\in(a_i,b_i)\}$$

is an event. Next, the class $\mathcal{B}$ is a σ-algebra over $\Re^k$ because (i) we have $\Re^k\in\mathcal{B}$ since $\{\boldsymbol{\xi}\in\Re^k\}=\Gamma$; (ii) if $B\in\mathcal{B}$, then $\{\boldsymbol{\xi}\in B\}$ is an event, and

$$\{\boldsymbol{\xi}\in B^c\}=\{\boldsymbol{\xi}\in B\}^c$$

is an event. This means that $B^c\in\mathcal{B}$; (iii) if $B_i\in\mathcal{B}$ for $i=1,2,\cdots$, then $\{\boldsymbol{\xi}\in B_i\}$ are events and

$$\left\{\boldsymbol{\xi}\in\bigcup_{i=1}^{\infty}B_i\right\}=\bigcup_{i=1}^{\infty}\{\boldsymbol{\xi}\in B_i\}$$

is an event. This means that $\cup_i B_i\in\mathcal{B}$. Since the smallest σ-algebra containing all open intervals of $\Re^k$ is just the Borel algebra over $\Re^k$, the class $\mathcal{B}$ contains all Borel sets of k-dimensional real vectors. The theorem is proved.

Definition 2.27 *(Liu [76]) The joint uncertainty distribution of an uncertain vector* $(\xi_1, \xi_2, \cdots, \xi_k)$ *is defined by*

$$\Phi(x_1, x_2, \cdots, x_k) = \mathcal{M}\{\xi_1 \le x_1, \xi_2 \le x_2, \cdots, \xi_k \le x_k\} \tag{2.226}$$

for any real numbers $x_1, x_2, \cdots, x_k$.

Theorem 2.58 *(Liu [76]) Let* $\xi_1, \xi_2, \cdots, \xi_k$ *be independent uncertain variables with uncertainty distributions* $\Phi_1, \Phi_2, \cdots, \Phi_k$*, respectively. Then the uncertain vector* $(\xi_1, \xi_2, \cdots, \xi_k)$ *has a joint uncertainty distribution*

$$\Phi(x_1, x_2, \cdots, x_k) = \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_k(x_k) \tag{2.227}$$

for any real numbers $x_1, x_2, \cdots, x_k$.

Proof: Since $\xi_1, \xi_2, \cdots, \xi_k$ are independent uncertain variables, we have

$$\Phi(x_1, x_2, \cdots, x_k) = \mathcal{M}\left\{\bigcap_{i=1}^{k}(\xi_i \le x_i)\right\} = \bigwedge_{i=1}^{k} \mathcal{M}\{\xi_i \le x_i\} = \bigwedge_{i=1}^{k} \Phi_i(x_i)$$

for any real numbers $x_1, x_2, \cdots, x_k$. The theorem is proved.

Remark 2.10: However, the equation (2.227) does not imply that the uncertain variables are independent. For example, let ξ be an uncertain variable with uncertainty distribution Φ. Then the joint uncertainty distribution Ψ of uncertain vector (ξ, ξ) is

$$\Psi(x_1, x_2) = \mathcal{M}\{(\xi \le x_1) \cap (\xi \le x_2)\} = \Phi(x_1) \wedge \Phi(x_2)$$

for any real numbers x_1 and x_2. But, generally speaking, an uncertain variable is not independent of itself.

Definition 2.28 *(Liu [91]) The k-dimensional uncertain vectors* $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots,$ $\boldsymbol{\xi}_n$ *are said to be independent if for any Borel sets* $B_1, B_2, \cdots, B_n$ *of k-dimensional real vectors, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\}. \tag{2.228}$$

Exercise 2.71: Let (ξ_1, ξ_2, ξ_3) and (η_1, η_2, η_3) be independent uncertain vectors. Show that ξ_1 and η_2 are independent uncertain variables.

Exercise 2.72: Let (ξ_1, ξ_2, ξ_3) and (η_1, η_2, η_3) be independent uncertain vectors. Show that (ξ_1, ξ_2) and (η_2, η_3) are independent uncertain vectors.

Theorem 2.59 *(Liu [91]) The k-dimensional uncertain vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\boldsymbol{\xi}_i \in B_i\right\} \tag{2.229}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of k-dimensional real vectors.

Proof: It follows from the duality of uncertain measure that $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are independent if and only if

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i^c)\right\}$$
$$= 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i^c\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\boldsymbol{\xi}_i \in B_i\right\}.$$

The theorem is thus proved.

Theorem 2.60 *Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ be independent uncertain vectors, and let $f_1, f_2, \cdots, f_n$ be vector-valued measurable functions. Then $f_1(\boldsymbol{\xi}_1), f_2(\boldsymbol{\xi}_2), \cdots, f_n(\boldsymbol{\xi}_n)$ are also independent uncertain vectors.*

Proof: For any Borel sets $B_1, B_2, \cdots, B_n$ of k-dimensional real vectors, it follows from the definition of independence that

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(f_i(\boldsymbol{\xi}_i) \in B_i)\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in f_i^{-1}(B_i))\right\}$$
$$= \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in f_i^{-1}(B_i)\} = \bigwedge_{i=1}^{n} \mathcal{M}\{f_i(\boldsymbol{\xi}_i) \in B_i\}.$$

Thus $f_1(\boldsymbol{\xi}_1), f_2(\boldsymbol{\xi}_2), \cdots, f_n(\boldsymbol{\xi}_n)$ are independent uncertain variables.

Normal Uncertain Vector

Definition 2.29 *(Liu [91]) Let $\tau_1, \tau_2, \cdots, \tau_m$ be independent normal uncertain variables with expected value 0 and variance 1. Then*

$$\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_m) \tag{2.230}$$

is called a standard normal uncertain vector.

It is easy to verify that a standard normal uncertain vector $(\tau_1, \tau_2, \cdots, \tau_m)$ has a joint uncertainty distribution

$$\Phi(x_1, x_2, \cdots, x_m) = \left(1 + \exp\left(-\frac{\pi(x_1 \wedge x_2 \wedge \cdots \wedge x_m)}{\sqrt{3}}\right)\right)^{-1} \tag{2.231}$$

for any real numbers $x_1, x_2, \cdots, x_m$. It is also easy to show that

$$\lim_{x_i \to -\infty} \Phi(x_1, x_2, \cdots, x_m) = 0, \quad \text{for each } i, \tag{2.232}$$

$$\lim_{(x_1, x_2, \cdots, x_m) \to +\infty} \Phi(x_1, x_2, \cdots, x_m) = 1. \tag{2.233}$$

Furthermore, the limit

$$\lim_{(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m) \to +\infty} \Phi(x_1, x_2, \cdots, x_m) \tag{2.234}$$

is a standard normal distribution with respect to x_i.

Definition 2.30 *(Liu [91]) Let $(\tau_1, \tau_2, \cdots, \tau_m)$ be a standard normal uncertain vector, and let e_i, σ_{ij}, $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, m$ be real numbers. Define*

$$\xi_i = e_i + \sum_{j=1}^{m} \sigma_{ij} \tau_j \tag{2.235}$$

for $i = 1, 2, \cdots, k$. Then $(\xi_1, \xi_2, \cdots, \xi_k)$ is called a normal uncertain vector.

That is, an uncertain vector $\boldsymbol{\xi}$ has a multivariate normal distribution if it can be represented in the form

$$\boldsymbol{\xi} = \boldsymbol{e} + \boldsymbol{\sigma}\boldsymbol{\tau} \tag{2.236}$$

for some real vector $\boldsymbol{e}$ and some real matrix $\boldsymbol{\sigma}$, where $\boldsymbol{\tau}$ is a standard normal uncertain vector. Note that $\boldsymbol{\xi}, \boldsymbol{e}$ and $\boldsymbol{\tau}$ are understood as column vectors. Please also note that for every index i, the component ξ_i is a normal uncertain variable with expected value e_i and standard variance

$$\sum_{j=1}^{m} |\sigma_{ij}|. \tag{2.237}$$

Theorem 2.61 *(Liu [91]) Assume $\boldsymbol{\xi}$ is a normal uncertain vector, $\boldsymbol{c}$ is a real vector, and D is a real matrix. Then*

$$\boldsymbol{\eta} = \boldsymbol{c} + D\boldsymbol{\xi} \tag{2.238}$$

is another normal uncertain vector.

Proof: Since $\boldsymbol{\xi}$ is a normal uncertain vector, there exists a standard normal uncertain vector $\boldsymbol{\tau}$, a real vector $\boldsymbol{e}$ and a real matrix $\boldsymbol{\sigma}$ such that $\boldsymbol{\xi} = \boldsymbol{e} + \boldsymbol{\sigma}\boldsymbol{\tau}$. It follows that

$$\boldsymbol{\eta} = \boldsymbol{c} + D\boldsymbol{\xi} = \boldsymbol{c} + D(\boldsymbol{e} + \boldsymbol{\sigma}\boldsymbol{\tau}) = (\boldsymbol{c} + D\boldsymbol{e}) + (D\boldsymbol{\sigma})\boldsymbol{\tau}.$$

Hence $\boldsymbol{\eta}$ is a normal uncertain vector.

2.15 Uncertain Matrix

This section introduces a concept of uncertain matrix that is a matrix all of whose elements are uncertain variables.

Definition 2.31 *(Liu [97]) A $p \times q$ uncertain matrix is a function $\boldsymbol{\xi}$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of $p \times q$ real matrices such that $\{\boldsymbol{\xi} \in B\}$ is an event for any Borel set B of $p \times q$ real matrices.*

Theorem 2.62 *(Liu [97]) The $p \times q$ matrix $\boldsymbol{\xi}$ is an uncertain matrix if and only if*

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1q} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{p1} & \xi_{p2} & \cdots & \xi_{pq} \end{pmatrix} \tag{2.239}$$

where $\xi_{ij}, i = 1, 2, \cdots, p, j = 1, 2, \cdots, q$ are uncertain variables.

Proof: Suppose that $\boldsymbol{\xi}$ is defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. For any Borel set B of real numbers, the set

$$B^* = \begin{pmatrix} B & \Re & \cdots & \Re \\ \Re & \Re & \cdots & \Re \\ \vdots & \vdots & \ddots & \vdots \\ \Re & \Re & \cdots & \Re \end{pmatrix}$$

is a Borel set of $p \times q$ real matrices. Thus the set $\{\xi_{11} \in B\} = \{\boldsymbol{\xi} \in B^*\}$ is an event. Hence ξ_{11} is an uncertain variable. A similar process may prove that other ξ_{ij}'s are uncertain variables.

Conversely, suppose that all ξ_{ij}'s are uncertain variables on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We define

$$\mathcal{B} = \left\{ B \subset \Re^{p \times q} \;\middle|\; \{\boldsymbol{\xi} \in B\} \text{ is an event} \right\}.$$

The matrix $\boldsymbol{\xi} = (\xi_{ij})_{p \times q}$ is proved to be an uncertain matrix if we can prove that $\mathcal{B}$ contains all Borel sets of $p \times q$ real matrices. First, the class $\mathcal{B}$ contains all open intervals of $\Re^{p \times q}$ because

$$\left\{ \boldsymbol{\xi} \in \begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & \cdots & (a_{1q}, b_{1q}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & \cdots & (a_{2q}, b_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{p1}, b_{p1}) & (a_{p2}, b_{p2}) & \cdots & (a_{pq}, b_{pq}) \end{pmatrix} \right\} = \bigcap_{i=1}^{p} \bigcap_{j=1}^{q} \{\xi_{ij} \in (a_{ij}, b_{ij})\}$$

is an event. Next, the class $\mathcal{B}$ is a σ-algebra over $\Re^{p\times q}$ because (i) we have $\Re^{p\times q} \in \mathcal{B}$ since $\{\boldsymbol{\xi} \in \Re^{p\times q}\} = \Gamma$; (ii) if $B \in \mathcal{B}$, then $\{\boldsymbol{\xi} \in B\}$ is an event, and

$$\{\boldsymbol{\xi} \in B^c\} = \{\boldsymbol{\xi} \in B\}^c$$

is an event. This means that $B^c \in \mathcal{B}$; (iii) if $B_i \in \mathcal{B}$ for $i = 1, 2, \cdots$, then $\{\boldsymbol{\xi} \in B_i\}$ are events and

$$\left\{\boldsymbol{\xi} \in \bigcup_{i=1}^{\infty} B_i\right\} = \bigcup_{i=1}^{\infty}\{\boldsymbol{\xi} \in B_i\}$$

is an event. This means that $\cup_i B_i \in \mathcal{B}$. Since the smallest σ-algebra containing all open intervals of $\Re^{p\times q}$ is just the Borel algebra over $\Re^{p\times q}$, the class $\mathcal{B}$ contains all Borel sets of $p \times q$ real matrices. The theorem is proved.

Definition 2.32 *(Liu [97]) The $p \times q$ uncertain matrices $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are said to be independent if for any Borel sets $B_1, B_2, \cdots, B_n$ of $p \times q$ real matrices, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\}. \tag{2.240}$$

Exercise 2.73: Let $(\xi_{ij})_{3\times 3}$ and $(\eta_{ij})_{3\times 3}$ be independent uncertain matrices. Show that (ξ_{11}, ξ_{12}) and $(\eta_{31}, \eta_{32}, \eta_{33})$ are independent uncertain vectors.

Exercise 2.74: Let $(\xi_{ij})_{3\times 3}$ and $(\eta_{ij})_{3\times 3}$ be independent uncertain matrices. Show that

$$\begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \end{pmatrix} \text{ and } \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \\ \eta_{31} & \eta_{32} \end{pmatrix}$$

are independent uncertain matrices.

Theorem 2.63 *(Liu [97]) The $p \times q$ uncertain matrices $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\boldsymbol{\xi}_i \in B_i\right\} \tag{2.241}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of $p \times q$ real matrices.

Proof: It follows from the duality of uncertain measure that $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are independent if and only if

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i^c)\right\}$$
$$= 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i^c\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\boldsymbol{\xi}_i \in B_i\right\}.$$

The theorem is thus proved.

Theorem 2.64 *(Liu [97]) Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ be independent uncertain matrices, and let $f_1, f_2, \cdots, f_n$ be matrix-valued measurable functions. Then $f_1(\boldsymbol{\xi}_1), f_2(\boldsymbol{\xi}_2), \cdots, f_n(\boldsymbol{\xi}_n)$ are also independent uncertain matrices.*

Proof: For any Borel sets $B_1, B_2, \cdots, B_n$ of real matrices, it follows from the definition of independence that

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(f_i(\boldsymbol{\xi}_i) \in B_i)\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in f_i^{-1}(B_i))\right\}$$

$$= \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in f_i^{-1}(B_i)\} = \bigwedge_{i=1}^{n} \mathcal{M}\{f_i(\boldsymbol{\xi}_i) \in B_i\}.$$

Thus $f_1(\boldsymbol{\xi}_1), f_2(\boldsymbol{\xi}_2), \cdots, f_n(\boldsymbol{\xi}_n)$ are independent uncertain variables.

2.16 Bibliographic Notes

As a fundamental concept in uncertainty theory, the uncertain variable was presented by Liu [76] in 2007. In order to describe uncertain variable, Liu [76] also introduced the uncertainty distribution. Later, Peng-Iwamura [120] proved a sufficient and necessary condition for uncertainty distribution. In addition, Liu [83] proposed the inverse uncertainty distribution, and Liu [88] verified a sufficient and necessary condition for it. Furthermore, Liu [76] proposed the conditional uncertainty distribution, and derived some formulas for calculating it.

Following the independence concept of uncertain variables proposed by Liu [79], the operational law was given by Liu [83] for calculating the uncertainty distribution and inverse uncertainty distribution of strictly monotone function of independent uncertain variables.

In order to rank uncertain variables, Liu [76] proposed the expected value operator. In addition, the linearity of expected value operator was verified by Liu [83]. As an important contribution, Liu-Ha [103] derived a useful formula for calculating the expected values of strictly monotone functions of independent uncertain variables. Based on the expected value operator, Liu [76] presented the variance, moments and distance between uncertain variables.

The entropy was proposed by Liu [79] for characterizing the uncertainty of uncertain variables. Chen-Dai [8] discussed the maximum entropy principle in order to select the uncertainty distribution that has maximum entropy and satisfies the prescribed constraints. Especially, normal uncertainty distribution is proved to have maximum entropy when the expected value and variance are fixed in advance.

Uncertain sequence was presented by Liu [76] with convergence almost surely, convergence in measure, convergence in mean, and convergence in distribution. Furthermore, Gao [40], You [187], Zhang [199], and Chen-Li-Ralescu [15] developed some other concepts of convergence and investigated their mathematical properties.

Uncertain vector was defined by Liu [76] as a measurable function from an uncertainty space to the set of real vectors. In addition, Liu [91] discussed the independence of uncertain vectors and proposed the concept of normal uncertain vector.

Uncertain matrix was suggested by Liu [97] as a measurable function from an uncertainty space to the set of real matrices.

Chapter 3

Uncertain Programming

Uncertain programming was founded by Liu [78] in 2009. This chapter will provide the theory of uncertain programming, and present some uncertain programming models for machine scheduling problem, vehicle routing problem, and project scheduling problem.

3.1 Uncertain Programming

Uncertain programming is a type of mathematical programming involving uncertain variables. Assume that $\boldsymbol{x}$ is a decision vector, and $\boldsymbol{\xi}$ is an uncertain vector. Since an uncertain objective function $f(\boldsymbol{x}, \boldsymbol{\xi})$ cannot be directly minimized, we may minimize its expected value, i.e.,

$$\min_{\boldsymbol{x}} E[f(\boldsymbol{x}, \boldsymbol{\xi})]. \tag{3.1}$$

In addition, since the uncertain constraints $g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0, j = 1, 2, \cdots, p$ do not define a crisp feasible set, it is naturally desired that the uncertain constraints hold with confidence levels $\alpha_1, \alpha_2, \cdots, \alpha_p$. Then we have a set of chance constraints,

$$\mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p. \tag{3.2}$$

In order to obtain a decision with minimum expected objective value subject to a set of chance constraints, Liu [78] proposed the following uncertain programming model,

$$\left\{\begin{array}{l} \min\limits_{\boldsymbol{x}} E[f(\boldsymbol{x}, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p. \end{array}\right. \tag{3.3}$$

Definition 3.1 *(Liu [78]) A vector $\boldsymbol{x}$ is called a feasible solution to the uncertain programming model (3.3) if*

$$\mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \alpha_j \tag{3.4}$$

for $j = 1, 2, \cdots, p$.

Definition 3.2 *(Liu [78]) A feasible solution $\boldsymbol{x}^*$ is called an optimal solution to the uncertain programming model (3.3) if*

$$E[f(\boldsymbol{x}^*, \boldsymbol{\xi})] \leq E[f(\boldsymbol{x}, \boldsymbol{\xi})] \tag{3.5}$$

for any feasible solution $\boldsymbol{x}$.

Theorem 3.1 *Assume the objective function $f(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$. If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the expected objective function $E[f(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)]$ is equal to*

$$\int_0^1 f(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \tag{3.6}$$

Proof: It follows from Theorem 2.26 immediately.

Exercise 3.1: Assume $f(\boldsymbol{x}, \boldsymbol{\xi}) = h_1(\boldsymbol{x})\xi_1 + h_2(\boldsymbol{x})\xi_2 + \cdots + h_n(\boldsymbol{x})\xi_n + h_0(\boldsymbol{x})$ where $h_1(\boldsymbol{x}), h_2(\boldsymbol{x}), \cdots, h_n(\boldsymbol{x}), h_0(\boldsymbol{x})$ are real-valued functions and $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables. Show that

$$E[f(\boldsymbol{x}, \boldsymbol{\xi})] = h_1(\boldsymbol{x})E[\xi_1] + h_2(\boldsymbol{x})E[\xi_2] + \cdots + h_n(\boldsymbol{x})E[\xi_n] + h_0(\boldsymbol{x}). \tag{3.7}$$

Theorem 3.2 *Assume the constraint function $g(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_k$ and strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \cdots, \xi_n$. If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the chance constraint*

$$\mathcal{M}\{g(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n) \leq 0\} \geq \alpha \tag{3.8}$$

holds if and only if

$$g(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \leq 0. \tag{3.9}$$

Proof: It follows from the operational law of uncertain variables that the inverse uncertainty distribution of $g(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)$ is

$$\Psi^{-1}(\alpha) = g(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Thus (3.8) holds if and only if $\Psi^{-1}(\alpha) \leq 0$. The theorem is thus verified.

Exercise 3.2: Assume $x_1, x_2, \cdots, x_n$ are nonnegative decision variables, and $\xi_1, \xi_2, \cdots, \xi_n, \xi$ are independent linear uncertain variables $\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2), \cdots, \mathcal{L}(a_n, b_n), \mathcal{L}(a, b)$, respectively. Show that for any confidence level $\alpha \in (0, 1)$, the chance constraint

$$\mathcal{M}\left\{\sum_{i=1}^{n} \xi_i x_i \le \xi\right\} \ge \alpha \tag{3.10}$$

holds if and only if

$$\sum_{i=1}^{n}((1-\alpha)a_i + \alpha b_i)x_i \le \alpha a + (1-\alpha)b. \tag{3.11}$$

Exercise 3.3: Assume $x_1, x_2, \cdots, x_n$ are nonnegative decision variables, and $\xi_1, \xi_2, \cdots, \xi_n, \xi$ are independent normal uncertain variables $\mathcal{N}(e_1, \sigma_1), \mathcal{N}(e_2, \sigma_2), \cdots, \mathcal{N}(e_n, \sigma_n), \mathcal{N}(e, \sigma)$, respectively. Show that for any confidence level $\alpha \in (0, 1)$, the chance constraint

$$\mathcal{M}\left\{\sum_{i=1}^{n} \xi_i x_i \le \xi\right\} \ge \alpha \tag{3.12}$$

holds if and only if

$$\sum_{i=1}^{n}\left(e_i + \frac{\sigma_i\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)x_i \le e - \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}. \tag{3.13}$$

Exercise 3.4: Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and $h_1(\boldsymbol{x})$, $h_2(\boldsymbol{x}), \cdots, h_n(\boldsymbol{x}), h_0(\boldsymbol{x})$ are real-valued functions. Show that

$$\mathcal{M}\left\{\sum_{i=1}^{n} h_i(\boldsymbol{x})\xi_i \le h_0(\boldsymbol{x})\right\} \ge \alpha \tag{3.14}$$

holds if and only if

$$\sum_{i=1}^{n} h_i^+(\boldsymbol{x})\Phi_i^{-1}(\alpha) - \sum_{i=1}^{n} h_i^-(\boldsymbol{x})\Phi_i^{-1}(1-\alpha) \le h_0(\boldsymbol{x}) \tag{3.15}$$

where

$$h_i^+(\boldsymbol{x}) = \begin{cases} h_i(\boldsymbol{x}), & \text{if } h_i(\boldsymbol{x}) > 0 \\ 0, & \text{if } h_i(\boldsymbol{x}) \le 0, \end{cases} \tag{3.16}$$

$$h_i^-(\boldsymbol{x}) = \begin{cases} -h_i(\boldsymbol{x}), & \text{if } h_i(\boldsymbol{x}) < 0 \\ 0, & \text{if } h_i(\boldsymbol{x}) \ge 0 \end{cases} \tag{3.17}$$

for $i = 1, 2, \cdots, n$.

Theorem 3.3 *Assume $f(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, and $g_j(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)$ are strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_k$ and strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \cdots, \xi_n$ for $j = 1, 2, \cdots, p$. If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the uncertain programming*

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[f(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)] \\ \textit{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n) \le 0\} \ge \alpha_j, \quad j = 1, 2, \cdots, p \end{cases} \tag{3.18}$$

is equivalent to the crisp mathematical programming

$$\begin{cases} \min\limits_{\boldsymbol{x}} \displaystyle\int_0^1 f(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \\ \textit{subject to:} \\ \qquad g_j(\boldsymbol{x}, \Phi_1^{-1}(\alpha_j), \cdots, \Phi_k^{-1}(\alpha_j), \Phi_{k+1}^{-1}(1-\alpha_j), \cdots, \Phi_n^{-1}(1-\alpha_j)) \le 0 \\ \qquad j = 1, 2, \cdots, p. \end{cases}$$

Proof: It follows from Theorems 3.1 and 3.2 immediately.

3.2 Numerical Method

When the objective functions and constraint functions are monotone with respect to the uncertain parameters, the uncertain programming model may be converted to a crisp mathematical programming.

It is fortunate for us that almost all objective and constraint functions in practical problems are indeed monotone with respect to the uncertain parameters (not decision variables).

From the mathematical viewpoint, there is no difference between crisp mathematical programming and classical mathematical programming except for an integral. Thus we may solve it by simplex method, branch-and-bound method, cutting plane method, implicit enumeration method, interior point method, gradient method, genetic algorithm, particle swarm optimization, neural networks, tabu search, and so on.

Example 3.1: Assume that x_1, x_2, x_3 are nonnegative decision variables, ξ_1, ξ_2, ξ_3 are independent linear uncertain variables $\mathcal{L}(1,2), \mathcal{L}(2,3), \mathcal{L}(3,4)$, and η_1, η_2, η_3 are independent zigzag uncertain variables $\mathcal{Z}(1,2,3), \mathcal{Z}(2,3,4), \mathcal{Z}(3,4,5)$, respectively. Consider the uncertain programming,

$$\begin{cases} \max\limits_{x_1, x_2, x_3} E\left[\sqrt{x_1 + \xi_1} + \sqrt{x_2 + \xi_2} + \sqrt{x_3 + \xi_3}\right] \\ \text{subject to:} \\ \qquad \mathcal{M}\{(x_1 + \eta_1)^2 + (x_2 + \eta_2)^2 + (x_3 + \eta_3)^2 \le 100\} \ge 0.9 \\ \qquad x_1, x_2, x_3 \ge 0. \end{cases}$$

Note that $\sqrt{x_1+\xi_1}+\sqrt{x_2+\xi_2}+\sqrt{x_3+\xi_3}$ is a strictly increasing function with respect to ξ_1,ξ_2,ξ_3, and $(x_1+\eta_1)^2+(x_2+\eta_2)^2+(x_3+\eta_3)^2$ is a strictly increasing function with respect to η_1,η_2,η_3. It is easy to verify that the uncertain programming model can be converted to the crisp model,

$$
\begin{cases}
\displaystyle\max_{x_1,x_2,x_3}\int_0^1\left(\sqrt{x_1+\Phi_1^{-1}(\alpha)}+\sqrt{x_2+\Phi_2^{-1}(\alpha)}+\sqrt{x_3+\Phi_3^{-1}(\alpha)}\right)\mathrm{d}\alpha\\
\text{subject to:}\\
\qquad (x_1+\Psi_1^{-1}(0.9))^2+(x_2+\Psi_2^{-1}(0.9))^2+(x_3+\Psi_3^{-1}(0.9))^2\le 100\\
\qquad x_1,x_2,x_3\ge 0
\end{cases}
$$

where $\Phi_1^{-1},\Phi_2^{-1},\Phi_3^{-1},\Psi_1^{-1},\Psi_2^{-1},\Psi_3^{-1}$ are inverse uncertainty distributions of uncertain variables $\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3$, respectively. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this model and obtain an optimal solution

$$(x_1^*,x_2^*,x_3^*)=(2.9735,1.9735,0.9735)$$

whose objective value is 6.3419.

Example 3.2: Assume that x_1 and x_2 are decision variables, ξ_1 and ξ_2 are iid linear uncertain variables $\mathcal{L}(0,\pi/2)$. Consider the uncertain programming,

$$
\begin{cases}
\displaystyle\min_{x_1,x_2} E\left[x_1\sin(x_1-\xi_1)-x_2\cos(x_2+\xi_2)\right]\\
\text{subject to:}\\
\qquad 0\le x_1\le\dfrac{\pi}{2},\quad 0\le x_2\le\dfrac{\pi}{2}.
\end{cases}
$$

It is clear that $x_1\sin(x_1-\xi_1)-x_2\cos(x_2+\xi_2)$ is strictly decreasing with respect to ξ_1 and strictly increasing with respect to ξ_2. Thus the uncertain programming is equivalent to the crisp model,

$$
\begin{cases}
\displaystyle\min_{x_1,x_2}\int_0^1\left(x_1\sin(x_1-\Phi_1^{-1}(1-\alpha))-x_2\cos(x_2+\Phi_2^{-1}(\alpha))\right)\mathrm{d}\alpha\\
\text{subject to:}\\
\qquad 0\le x_1\le\dfrac{\pi}{2},\quad 0\le x_2\le\dfrac{\pi}{2}
\end{cases}
$$

where Φ_1^{-1},Φ_2^{-1} are inverse uncertainty distributions of ξ_1,ξ_2, respectively. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this model and obtain an optimal solution

$$(x_1^*,x_2^*)=(0.4026,0.4026)$$

whose objective value is -0.2708.

3.3 Machine Scheduling Problem

Machine scheduling problem is concerned with finding an efficient schedule during an uninterrupted period of time for a set of machines to process a set of jobs. A lot of research work has been done on this type of problem. The study of machine scheduling problem with uncertain processing times was started by Liu [83] in 2010.

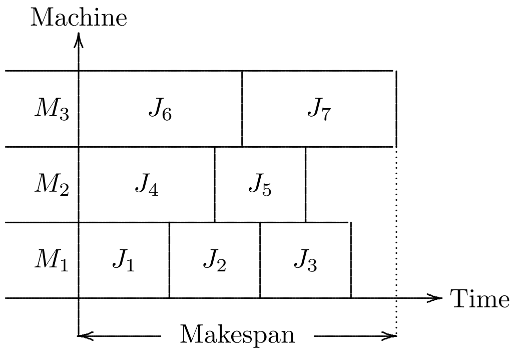


Figure 3.1: A Machine Schedule with 3 Machines and 7 Jobs

In a machine scheduling problem, we assume that (a) each job can be processed on any machine without interruption; (b) each machine can process only one job at a time; and (c) the processing times are uncertain variables with known uncertainty distributions. We also use the following indices and parameters:

$i = 1, 2, \cdots, n$: jobs;

$k = 1, 2, \cdots, m$: machines;

ξ_{ik}: uncertain processing time of job i on machine k;

Φ_{ik}: uncertainty distribution of ξ_{ik}.

How to Represent a Schedule?

Liu [74] suggested that a schedule should be represented by two decision vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, where

$\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$: integer decision vector representing n jobs with $1 \le x_i \le n$ and $x_i \ne x_j$ for all $i \ne j$, $i, j = 1, 2, \cdots, n$. That is, the sequence $\{x_1, x_2, \cdots, x_n\}$ is a rearrangement of $\{1, 2, \cdots, n\}$;

$\boldsymbol{y} = (y_1, y_2, \cdots, y_{m-1})$: integer decision vector with $y_0 \equiv 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \equiv y_m$.

We note that the schedule is fully determined by the decision vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in the following way. For each k ($1 \le k \le m$), if $y_k = y_{k-1}$, then the machine k is not used; if $y_k > y_{k-1}$, then the machine k is used and processes jobs $x_{y_{k-1}+1}, x_{y_{k-1}+2}, \cdots, x_{y_k}$ in turn. Thus the schedule of all machines is as follows,

$$\begin{aligned}
&\text{Machine 1: } x_{y_0+1} \to x_{y_0+2} \to \cdots \to x_{y_1};\\
&\text{Machine 2: } x_{y_1+1} \to x_{y_1+2} \to \cdots \to x_{y_2};\\
&\quad\cdots\\
&\text{Machine } m\text{: } x_{y_{m-1}+1} \to x_{y_{m-1}+2} \to \cdots \to x_{y_m}.
\end{aligned} \tag{3.19}$$

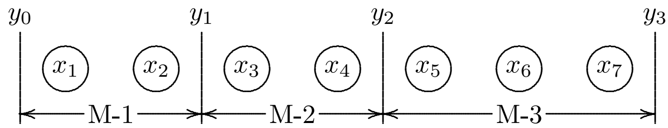


Figure 3.2: Formulation of Schedule in which Machine 1 processes Jobs x_1, x_2, Machine 2 processes Jobs x_3, x_4 and Machine 3 processes Jobs x_5, x_6, x_7.

Completion Times

Let $C_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ be the completion times of jobs i, $i = 1, 2, \cdots, n$, respectively. For each k with $1 \le k \le m$, if the machine k is used (i.e., $y_k > y_{k-1}$), then we have

$$C_{x_{y_{k-1}+1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) = \xi_{x_{y_{k-1}+1}k} \tag{3.20}$$

and

$$C_{x_{y_{k-1}+j}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) = C_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) + \xi_{x_{y_{k-1}+j}k} \tag{3.21}$$

for $2 \le j \le y_k - y_{k-1}$.

If the machine k is used, then the completion time $C_{x_{y_{k-1}+1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ of job $x_{y_{k-1}+1}$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}_{x_{y_{k-1}+1}}(\boldsymbol{x}, \boldsymbol{y}, \alpha) = \Phi^{-1}_{x_{y_{k-1}+1}k}(\alpha). \tag{3.22}$$

Generally, suppose the completion time $C_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ has an inverse uncertainty distribution $\Psi^{-1}_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \alpha)$. Then the completion time $C_{x_{y_{k-1}+j}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ has an inverse uncertainty distribution

$$\Psi^{-1}_{x_{y_{k-1}+j}}(\boldsymbol{x}, \boldsymbol{y}, \alpha) = \Psi^{-1}_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \alpha) + \Phi^{-1}_{x_{y_{k-1}+j}k}(\alpha). \tag{3.23}$$

This recursive process may produce all inverse uncertainty distributions of completion times of jobs.

Makespan

Note that, for each k $(1 \le k \le m)$, the value $C_{x_{y_k}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi})$ is just the time that the machine k finishes all jobs assigned to it. Thus the makespan of the schedule $(\boldsymbol{x},\boldsymbol{y})$ is determined by

$$f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi}) = \max_{1\le k\le m} C_{x_{y_k}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi}) \tag{3.24}$$

whose inverse uncertainty distribution is

$$\Upsilon^{-1}(\boldsymbol{x},\boldsymbol{y},\alpha) = \max_{1\le k\le m} \Psi^{-1}_{x_{y_k}}(\boldsymbol{x},\boldsymbol{y},\alpha). \tag{3.25}$$

Machine Scheduling Model

In order to minimize the expected makespan $E[f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi})]$, we have the following machine scheduling model,

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y}} E[f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad 1 \le x_i \le n, \quad i = 1,2,\cdots,n \\ \qquad x_i \ne x_j, \quad i \ne j,\ i,j = 1,2,\cdots,n \\ \qquad 0 \le y_1 \le y_2 \cdots \le y_{m-1} \le n \\ \qquad x_i, y_j, \quad i = 1,2,\cdots,n, \quad j = 1,2,\cdots,m-1, \quad \text{integers.} \end{cases} \tag{3.26}$$

Since $\Upsilon^{-1}(\boldsymbol{x},\boldsymbol{y},\alpha)$ is the inverse uncertainty distribution of $f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi})$, the machine scheduling model is simplified as follows,

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y}} \displaystyle\int_0^1 \Upsilon^{-1}(\boldsymbol{x},\boldsymbol{y},\alpha)\mathrm{d}\alpha \\ \text{subject to:} \\ \qquad 1 \le x_i \le n, \quad i = 1,2,\cdots,n \\ \qquad x_i \ne x_j, \quad i \ne j,\ i,j = 1,2,\cdots,n \\ \qquad 0 \le y_1 \le y_2 \cdots \le y_{m-1} \le n \\ \qquad x_i, y_j, \quad i = 1,2,\cdots,n, \quad j = 1,2,\cdots,m-1, \quad \text{integers.} \end{cases} \tag{3.27}$$

Numerical Experiment

Assume that there are 3 machines and 7 jobs with the following linear uncertain processing times

$$\xi_{ik} \sim \mathcal{L}(i, i+k), \quad i = 1,2,\cdots,7,\ k = 1,2,3$$

where i is the index of jobs and k is the index of machines. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields that the optimal solution is

$$\boldsymbol{x}^* = (1, 4, 5, 3, 7, 2, 6), \quad \boldsymbol{y}^* = (3, 5). \tag{3.28}$$

In other words, the optimal machine schedule is

Machine 1: $1 \to 4 \to 5$
Machine 2: $3 \to 7$
Machine 3: $2 \to 6$

whose expected makespan is 12.

3.4 Vehicle Routing Problem

Vehicle routing problem (VRP) is concerned with finding efficient routes, beginning and ending at a central depot, for a fleet of vehicles to serve a number of customers.

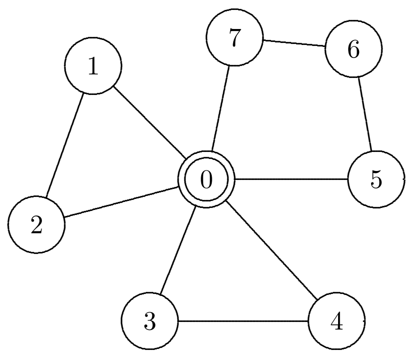


Figure 3.3: A Vehicle Routing Plan with Single Depot and 7 Customers

Due to its wide applicability and economic importance, vehicle routing problem has been extensively studied. Liu [83] first introduced uncertainty theory into the research area of vehicle routing problem in 2010. In this section, vehicle routing problem will be modelled by uncertain programming in which the travel times are assumed to be uncertain variables with known uncertainty distributions.

We assume that (a) a vehicle will be assigned for only one route on which there may be more than one customer; (b) a customer will be visited by one and only one vehicle; (c) each route begins and ends at the depot; and (d) each customer specifies its time window within which the delivery is permitted or preferred to start.

Let us first introduce the following indices and model parameters:
$i = 0$: depot;
$i = 1, 2, \cdots, n$: customers;

$k = 1, 2, \cdots, m$: vehicles;
D_{ij}: travel distance from customers i to j, $i, j = 0, 1, 2, \cdots, n$;
T_{ij}: uncertain travel time from customers i to j, $i, j = 0, 1, 2, \cdots, n$;
Φ_{ij}: uncertainty distribution of T_{ij}, $i, j = 0, 1, 2, \cdots, n$;
$[a_i, b_i]$: time window of customer i, $i = 1, 2, \cdots, n$.

Operational Plan

Liu [74] suggested that an operational plan should be represented by three decision vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{t}$, where

$\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$: integer decision vector representing n customers with $1 \le x_i \le n$ and $x_i \ne x_j$ for all $i \ne j$, $i, j = 1, 2, \cdots, n$. That is, the sequence $\{x_1, x_2, \cdots, x_n\}$ is a rearrangement of $\{1, 2, \cdots, n\}$;

$\boldsymbol{y} = (y_1, y_2, \cdots, y_{m-1})$: integer decision vector with $y_0 \equiv 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \equiv y_m$;

$\boldsymbol{t} = (t_1, t_2, \cdots, t_m)$: each t_k represents the starting time of vehicle k at the depot, $k = 1, 2, \cdots, m$.

We note that the operational plan is fully determined by the decision vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{t}$ in the following way. For each k ($1 \le k \le m$), if $y_k = y_{k-1}$, then vehicle k is not used; if $y_k > y_{k-1}$, then vehicle k is used and starts from the depot at time t_k, and the tour of vehicle k is $0 \to x_{y_{k-1}+1} \to x_{y_{k-1}+2} \to \cdots \to x_{y_k} \to 0$. Thus the tours of all vehicles are as follows:

$$
\begin{aligned}
&\text{Vehicle 1: } 0 \to x_{y_0+1} \to x_{y_0+2} \to \cdots \to x_{y_1} \to 0; \\
&\text{Vehicle 2: } 0 \to x_{y_1+1} \to x_{y_1+2} \to \cdots \to x_{y_2} \to 0; \\
&\qquad \cdots \\
&\text{Vehicle } m\text{: } 0 \to x_{y_{m-1}+1} \to x_{y_{m-1}+2} \to \cdots \to x_{y_m} \to 0.
\end{aligned}
$$

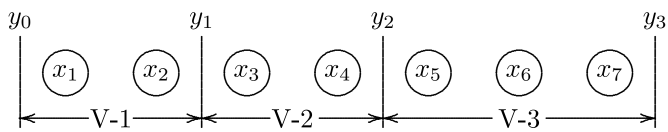


Figure 3.4: Formulation of Operational Plan in which Vehicle 1 visits Customers x_1, x_2, Vehicle 2 visits Customers x_3, x_4 and Vehicle 3 visits Customers x_5, x_6, x_7.

It is clear that this type of representation is intuitive, and the total number of decision variables is $n + 2m - 1$. We also note that the above decision variables $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{t}$ ensure that: (a) each vehicle will be used at most one time; (b) all tours begin and end at the depot; (c) each customer will be visited by one and only one vehicle; and (d) there is no subtour.

Arrival Times

Let $f_i(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ be the arrival time function of some vehicles at customers i for $i=1,2,\cdots,n$. We remind readers that $f_i(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ are determined by the decision variables $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{t}$, $i=1,2,\cdots,n$. Since unloading can start either immediately, or later, when a vehicle arrives at a customer, the calculation of $f_i(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ is heavily dependent on the operational strategy. Here we assume that the customer does not permit a delivery earlier than the time window. That is, the vehicle will wait to unload until the beginning of the time window if it arrives before the time window. If a vehicle arrives at a customer after the beginning of the time window, unloading will start immediately. For each k with $1\le k\le m$, if vehicle k is used (i.e., $y_k>y_{k-1}$), then we have

$$f_{x_{y_{k-1}+1}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})=t_k+T_{0x_{y_{k-1}+1}}$$

and

$$f_{x_{y_{k-1}+j}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})=f_{x_{y_{k-1}+j-1}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})\vee a_{x_{y_{k-1}+j-1}}+T_{x_{y_{k-1}+j-1}x_{y_{k-1}+j}}$$

for $2\le j\le y_k-y_{k-1}$. If the vehicle k is used, i.e., $y_k>y_{k-1}$, then the arrival time $f_{x_{y_{k-1}+1}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ at the customer $x_{y_{k-1}+1}$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}_{x_{y_{k-1}+1}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t},\alpha)=t_k+\Phi^{-1}_{0x_{y_{k-1}+1}}(\alpha).$$

Generally, suppose the arrival time $f_{x_{y_{k-1}+j-1}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ has an inverse uncertainty distribution $\Psi^{-1}_{x_{y_{k-1}+j-1}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t},\alpha)$. Then $f_{x_{y_{k-1}+j}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ has an inverse uncertainty distribution

$$\Psi^{-1}_{x_{y_{k-1}+j}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t},\alpha)=\Psi^{-1}_{x_{y_{k-1}+j-1}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t},\alpha)\vee a_{x_{y_{k-1}+j-1}}+\Phi^{-1}_{x_{y_{k-1}+j-1}x_{y_{k-1}+j}}(\alpha)$$

for $2\le j\le y_k-y_{k-1}$. This recursive process may produce all inverse uncertainty distributions of arrival times at customers.

Travel Distance

Let $g(\boldsymbol{x},\boldsymbol{y})$ be the total travel distance of all vehicles. Then we have

$$g(\boldsymbol{x},\boldsymbol{y})=\sum_{k=1}^{m}g_k(\boldsymbol{x},\boldsymbol{y}) \tag{3.29}$$

where

$$g_k(\boldsymbol{x},\boldsymbol{y})=\begin{cases} D_{0x_{y_{k-1}+1}}+\sum\limits_{j=y_{k-1}+1}^{y_k-1}D_{x_jx_{j+1}}+D_{x_{y_k}0}, & \text{if } y_k>y_{k-1}\\ 0, & \text{if } y_k=y_{k-1}\end{cases}$$

for $k=1,2,\cdots,m$.

Vehicle Routing Model

If we hope that each customer i ($1 \le i \le n$) is visited within its time window $[a_i, b_i]$ with confidence level α_i (i.e., the vehicle arrives at customer i before time b_i), then we have the following chance constraint,

$$\mathcal{M}\{f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \le b_i\} \ge \alpha_i. \tag{3.30}$$

If we want to minimize the total travel distance of all vehicles subject to the time window constraint, then we have the following vehicle routing model,

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}} g(\boldsymbol{x}, \boldsymbol{y}) \\ \text{subject to:} \\ \qquad \mathcal{M}\{f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \le b_i\} \ge \alpha_i, \quad i = 1, 2, \cdots, n \\ \qquad 1 \le x_i \le n, \quad i = 1, 2, \cdots, n \\ \qquad x_i \ne x_j, \quad i \ne j, \ i, j = 1, 2, \cdots, n \\ \qquad 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \\ \qquad x_i, y_j, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1, \quad \text{integers} \end{cases} \tag{3.31}$$

which is equivalent to

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}} g(\boldsymbol{x}, \boldsymbol{y}) \\ \text{subject to:} \\ \qquad \Psi_i^{-1}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}, \alpha_i) \le b_i, \quad i = 1, 2, \cdots, n \\ \qquad 1 \le x_i \le n, \quad i = 1, 2, \cdots, n \\ \qquad x_i \ne x_j, \quad i \ne j, \ i, j = 1, 2, \cdots, n \\ \qquad 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \\ \qquad x_i, y_j, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1, \quad \text{integers} \end{cases} \tag{3.32}$$

where $\Psi_i^{-1}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}, \alpha)$ are the inverse uncertainty distributions of $f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ for $i = 1, 2, \cdots, n$, respectively.

Numerical Experiment

Assume that there are 3 vehicles and 7 customers with time windows shown in Table 3.1, and each customer is visited within time windows with confidence level 0.90.

We also assume that the distances are $D_{ij} = |i-j|$ for $i, j = 0, 1, 2, \cdots, 7$, and the travel times are normal uncertain variables

$$T_{ij} \sim \mathcal{N}(2|i-j|, 1), \quad i, j = 0, 1, 2, \cdots, 7.$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may

Table 3.1: Time Windows of Customers

Node	Window	Node	Window
1	$[7:00, 9:00]$	5	$[15:00, 17:00]$
2	$[7:00, 9:00]$	6	$[19:00, 21:00]$
3	$[15:00, 17:00]$	7	$[19:00, 21:00]$
4	$[15:00, 17:00]$		

yield that the optimal solution is

$$\begin{aligned} \boldsymbol{x}^* &= (1, 3, 2, 5, 7, 4, 6), \\ \boldsymbol{y}^* &= (2, 5), \\ \boldsymbol{t}^* &= (6:18, 4:18, 8:18). \end{aligned} \tag{3.33}$$

In other words, the optimal operational plan is

Vehicle 1: depot $\to 1 \to 3 \to$ depot (the latest starting time is 6:18)
Vehicle 2: depot $\to 2 \to 5 \to 7 \to$ depot (the latest starting time is 4:18)
Vehicle 3: depot $\to 4 \to 6 \to$ depot (the latest starting time is 8:18)

whose total travel distance is 32.

3.5 Project Scheduling Problem

Project scheduling problem is to determine the schedule of allocating resources so as to balance the total cost and the completion time. The study of project scheduling problem with uncertain factors was started by Liu [83] in 2010. This section presents an uncertain programming model for project scheduling problem in which the duration times are assumed to be uncertain variables with known uncertainty distributions.

Project scheduling is usually represented by a directed acyclic network where nodes correspond to milestones, and arcs to activities which are basically characterized by the times and costs consumed.

Let $(\mathcal{V}, \mathcal{A})$ be a directed acyclic graph, where $\mathcal{V} = \{1, 2, \cdots, n, n+1\}$ is the set of nodes, $\mathcal{A}$ is the set of arcs, $(i, j) \in \mathcal{A}$ is the arc of the graph $(\mathcal{V}, \mathcal{A})$ from nodes i to j. It is well-known that we can rearrange the indexes of the nodes in $\mathcal{V}$ such that $i < j$ for all $(i, j) \in \mathcal{A}$.

Before we begin to study project scheduling problem with uncertain activity duration times, we first make some assumptions: (a) all of the costs needed are obtained via loans with some given interest rate; and (b) each activity can be processed only if the loan needed is allocated and all the foregoing activities are finished.

In order to model the project scheduling problem, we introduce the following indices and parameters:

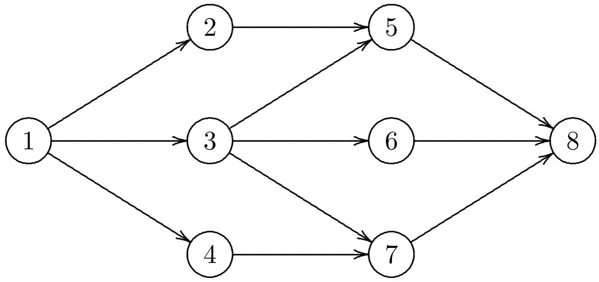


Figure 3.5: A Project with 8 Milestones and 11 Activities

ξ_{ij}: uncertain duration time of activity (i,j) in $\mathcal{A}$;
Φ_{ij}: uncertainty distribution of ξ_{ij};
c_{ij}: cost of activity (i,j) in $\mathcal{A}$;
r: interest rate;
x_i: integer decision variable representing the allocating time of all loans needed for all activities (i,j) in $\mathcal{A}$.

Starting Times

For simplicity, we write $\boldsymbol{\xi} = \{\xi_{ij} : (i,j) \in \mathcal{A}\}$ and $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$. Let $T_i(\boldsymbol{x}, \boldsymbol{\xi})$ denote the starting time of all activities (i,j) in $\mathcal{A}$. According to the assumptions, the starting time of the total project (i.e., the starting time of of all activities $(1,j)$ in $\mathcal{A}$) should be

$$T_1(\boldsymbol{x}, \boldsymbol{\xi}) = x_1 \tag{3.34}$$

whose inverse uncertainty distribution may be written as

$$\Psi_1^{-1}(\boldsymbol{x}, \alpha) = x_1. \tag{3.35}$$

From the starting time $T_1(\boldsymbol{x}, \boldsymbol{\xi})$, we deduce that the starting time of activity $(2,5)$ is

$$T_2(\boldsymbol{x}, \boldsymbol{\xi}) = x_2 \vee (x_1 + \xi_{12}) \tag{3.36}$$

whose inverse uncertainty distribution may be written as

$$\Psi_2^{-1}(\boldsymbol{x}, \alpha) = x_2 \vee (x_1 + \Phi_{12}^{-1}(\alpha)). \tag{3.37}$$

Generally, suppose that the starting time $T_k(\boldsymbol{x}, \boldsymbol{\xi})$ of all activities (k,i) in $\mathcal{A}$ has an inverse uncertainty distribution $\Psi_k^{-1}(\boldsymbol{x}, \alpha)$. Then the starting time $T_i(\boldsymbol{x}, \boldsymbol{\xi})$ of all activities (i,j) in $\mathcal{A}$ should be

$$T_i(\boldsymbol{x}, \boldsymbol{\xi}) = x_i \vee \max_{(k,i)\in\mathcal{A}} (T_k(\boldsymbol{x}, \boldsymbol{\xi}) + \xi_{ki}) \tag{3.38}$$

whose inverse uncertainty distribution is

$$\Psi_i^{-1}(\boldsymbol{x}, \alpha) = x_i \vee \max_{(k,i)\in\mathcal{A}} \left(\Psi_k^{-1}(\boldsymbol{x}, \alpha) + \Phi_{ki}^{-1}(\alpha)\right). \tag{3.39}$$

This recursive process may produce all inverse uncertainty distributions of starting times of activities.

Completion Time

The completion time $T(\boldsymbol{x}, \boldsymbol{\xi})$ of the total project (i.e, the finish time of all activities $(k, n+1)$ in $\mathcal{A}$) is

$$T(\boldsymbol{x}, \boldsymbol{\xi}) = \max_{(k,n+1)\in\mathcal{A}} \left(T_k(\boldsymbol{x}, \boldsymbol{\xi}) + \xi_{k,n+1}\right) \tag{3.40}$$

whose inverse uncertainty distribution is

$$\Psi^{-1}(\boldsymbol{x}, \alpha) = \max_{(k,n+1)\in\mathcal{A}} \left(\Psi_k^{-1}(\boldsymbol{x}, \alpha) + \Phi_{k,n+1}^{-1}(\alpha)\right). \tag{3.41}$$

Total Cost

Based on the completion time $T(\boldsymbol{x}, \boldsymbol{\xi})$, the total cost of the project can be written as

$$C(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{(i,j)\in\mathcal{A}} c_{ij}\,(1+r)^{\lceil T(\boldsymbol{x},\boldsymbol{\xi}) - x_i\rceil} \tag{3.42}$$

where $\lceil a\rceil$ represents the minimal integer greater than or equal to a. Note that $C(\boldsymbol{x}, \boldsymbol{\xi})$ is a discrete uncertain variable whose inverse uncertainty distribution is

$$\Upsilon^{-1}(\boldsymbol{x}, \alpha) = \sum_{(i,j)\in\mathcal{A}} c_{ij}\,(1+r)^{\lceil \Psi^{-1}(\boldsymbol{x};\alpha) - x_i\rceil} \tag{3.43}$$

for $0 < \alpha < 1$.

Project Scheduling Model

In order to minimize the expected cost of the project under the completion time constraint, we may construct the following project scheduling model,

$$\begin{cases} \displaystyle\min_{\boldsymbol{x}} E[C(\boldsymbol{x}, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{T(\boldsymbol{x}, \boldsymbol{\xi}) \le T_0\} \ge \alpha_0 \\ \qquad \boldsymbol{x} \ge 0, \text{ integer vector} \end{cases} \tag{3.44}$$

where T_0 is a due date of the project, α_0 is a predetermined confidence level, $T(\boldsymbol{x}, \boldsymbol{\xi})$ is the completion time defined by (3.40), and $C(\boldsymbol{x}, \boldsymbol{\xi})$ is the total cost defined by (3.42). This model is equivalent to

$$\left\{\begin{array}{l} \min\limits_{\boldsymbol{x}} \displaystyle\int_0^1 \Upsilon^{-1}(\boldsymbol{x},\alpha)\mathrm{d}\alpha \\ \text{subject to:} \\ \qquad \Psi^{-1}(\boldsymbol{x},\alpha_0) \le T_0 \\ \qquad \boldsymbol{x} \ge 0, \text{ integer vector} \end{array}\right. \tag{3.45}$$

where $\Psi^{-1}(\boldsymbol{x},\alpha)$ is the inverse uncertainty distribution of $T(\boldsymbol{x},\boldsymbol{\xi})$ determined by (3.41) and $\Upsilon^{-1}(\boldsymbol{x},\alpha)$ is the inverse uncertainty distribution of $C(\boldsymbol{x},\boldsymbol{\xi})$ determined by (3.43).

Numerical Experiment

Consider a project scheduling problem shown by Figure 3.5 in which there are 8 milestones and 11 activities. Assume that all duration times of activities are linear uncertain variables,

$$\xi_{ij} \sim \mathcal{L}(3i, 3j), \quad \forall (i,j) \in \mathcal{A}$$

and the costs of activities are

$$c_{ij} = i + j, \quad \forall (i,j) \in \mathcal{A}.$$

In addition, we also suppose that the interest rate is $r = 0.02$, the due date is $T_0 = 60$, and the confidence level is $\alpha_0 = 0.85$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields that the optimal solution is

$$\boldsymbol{x}^* = (7, 24, 17, 16, 35, 33, 30). \tag{3.46}$$

In other words, the optimal allocating times of all loans needed for all activities are shown in Table 3.2 whose expected total cost is 190.6, and

$$\mathcal{M}\{T(\boldsymbol{x}^*,\boldsymbol{\xi}) \le 60\} = 0.88.$$

Table 3.2: Optimal Allocating Times of Loans

Date	7	16	17	24	30	33	35
Node	1	4	3	2	7	6	5
Loan	12	11	27	7	15	14	13

3.6 Uncertain Multiobjective Programming

It has been increasingly recognized that many real decision-making problems involve multiple, noncommensurable, and conflicting objectives which should be considered simultaneously. In order to optimize multiple objectives, multiobjective programming has been well developed and applied widely. For modelling multiobjective decision-making problems with uncertain parameters, Liu-Chen [95] presented the following uncertain multiobjective programming,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \left(E[f_1(\boldsymbol{x},\boldsymbol{\xi})], E[f_2(\boldsymbol{x},\boldsymbol{\xi})], \cdots, E[f_m(\boldsymbol{x},\boldsymbol{\xi})]\right) \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1,2,\cdots,p \end{cases} \tag{3.47}$$

where $f_i(\boldsymbol{x},\boldsymbol{\xi})$ are objective functions for $i = 1,2,\cdots,m$, $g_j(\boldsymbol{x},\boldsymbol{\xi})$ are constraint functions, and α_j are confidence levels for $j = 1,2,\cdots,p$.

Since the objectives are usually in conflict, there is no optimal solution that simultaneously minimizes all the objective functions. In this case, we have to introduce the concept of *Pareto solution*, which means that it is impossible to improve any one objective without sacrificing on one or more of the other objectives.

Definition 3.3 *A feasible solution $\boldsymbol{x}^*$ is said to be Pareto to the uncertain multiobjective programming (3.47) if there is no feasible solution $\boldsymbol{x}$ such that*

$$E[f_i(\boldsymbol{x},\boldsymbol{\xi})] \le E[f_i(\boldsymbol{x}^*,\boldsymbol{\xi})], \quad i = 1,2,\cdots,m \tag{3.48}$$

and $E[f_j(\boldsymbol{x},\boldsymbol{\xi})] < E[f_j(\boldsymbol{x}^*,\boldsymbol{\xi})]$ for at least one index j.

If the decision maker has a real-valued *preference function* aggregating the m objective functions, then we may minimize the aggregating preference function subject to the same set of chance constraints. This model is referred to as a *compromise model* whose solution is called a *compromise solution*. It has been proved that the compromise solution is Pareto to the original multiobjective model.

The first well-known compromise model is set up by weighting the objective functions, i.e.,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \sum\limits_{i=1}^{m} \lambda_i E[f_i(\boldsymbol{x},\boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1,2,\cdots,p \end{cases} \tag{3.49}$$

where the weights $\lambda_1, \lambda_2, \cdots, \lambda_m$ are nonnegative numbers with $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, for example, $\lambda_i \equiv 1/m$ for $i = 1,2,\cdots,m$.

The second way is related to minimizing the *distance function* from a solution

$$\left(E[f_1(\boldsymbol{x},\boldsymbol{\xi})], E[f_2(\boldsymbol{x},\boldsymbol{\xi})], \cdots, E[f_m(\boldsymbol{x},\boldsymbol{\xi})]\right) \tag{3.50}$$

to an ideal vector $(f_1^*, f_2^*, \cdots, f_m^*)$, where f_i^* are the optimal values of the ith objective functions without considering other objectives, $i = 1, 2, \cdots, m$, respectively. That is,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \sum\limits_{i=1}^{m} \lambda_i (E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] - f_i^*)^2 \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1, 2, \cdots, p \end{cases} \tag{3.51}$$

where the weights $\lambda_1, \lambda_2, \cdots, \lambda_m$ are nonnegative numbers with $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, for example, $\lambda_i \equiv 1/m$ for $i = 1, 2, \cdots, m$.

By the third way a compromise solution can be found via an *interactive approach* consisting of a sequence of decision phases and computation phases. Various interactive approaches have been developed.

3.7 Uncertain Goal Programming

The concept of goal programming was presented by Charnes-Cooper [4] in 1961 and subsequently studied by many researchers. Goal programming can be regarded as a special compromise model for multiobjective optimization and has been applied in a wide variety of real-world problems. In multiobjective decision-making problems, we assume that the decision-maker is able to assign a target level for each goal and the key idea is to minimize the deviations (positive, negative, or both) from the target levels. In the real-world situation, the goals are achievable only at the expense of other goals and these goals are usually incompatible. In order to balance multiple conflicting objectives, a decision-maker may establish a hierarchy of importance among these incompatible goals so as to satisfy as many goals as possible in the order specified. For multiobjective decision-making problems with uncertain parameters, Liu-Chen [95] proposed an uncertain goal programming,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \sum\limits_{j=1}^{l} P_j \sum\limits_{i=1}^{m} (u_{ij} d_i^+ + v_{ij} d_i^-) \\ \text{subject to:} \\ \qquad E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] + d_i^- - d_i^+ = b_i, \quad i = 1, 2, \cdots, m \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j, \qquad j = 1, 2, \cdots, p \\ \qquad d_i^+, d_i^- \ge 0, \qquad\qquad\qquad i = 1, 2, \cdots, m \end{cases} \tag{3.52}$$

where P_j are the preemptive priority factors, u_{ij} and v_{ij} are the weighting factors, d_i^+ are the positive deviations, d_i^- are the negative deviations, f_i are the functions in goal constraints, g_j are the functions in real constraints, b_i are the target values, α_j are the confidence levels, l is the number of priorities, m is the number of goal constraints, and p is the number of real constraints.

Note that the positive and negative deviations are calculated by

$$d_i^+ = \begin{cases} E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] - b_i, & \text{if } E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] > b_i \\ 0, & \text{otherwise} \end{cases} \tag{3.53}$$

and

$$d_i^- = \begin{cases} b_i - E[f_i(\boldsymbol{x}, \boldsymbol{\xi})], & \text{if } E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] < b_i \\ 0, & \text{otherwise} \end{cases} \tag{3.54}$$

for each i. Sometimes, the objective function in the goal programming model is written as follows,

$$\text{lexmin}\left\{\sum_{i=1}^{m}(u_{i1}d_i^+ + v_{i1}d_i^-), \sum_{i=1}^{m}(u_{i2}d_i^+ + v_{i2}d_i^-), \cdots, \sum_{i=1}^{m}(u_{il}d_i^+ + v_{il}d_i^-)\right\}$$

where lexmin represents lexicographically minimizing the objective vector.

3.8 Uncertain Multilevel Programming

Multilevel programming offers a means of studying decentralized decision systems in which we assume that the leader and followers may have their own decision variables and objective functions, and the leader can only influence the reactions of followers through his own decision variables, while the followers have full authority to decide how to optimize their own objective functions in view of the decisions of the leader and other followers.

Assume that in a decentralized two-level decision system there is one leader and m followers. Let $\boldsymbol{x}$ and $\boldsymbol{y}_i$ be the control vectors of the leader and the ith followers, $i = 1, 2, \cdots, m$, respectively. We also assume that the objective functions of the leader and ith followers are $F(\boldsymbol{x}, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi})$ and $f_i(\boldsymbol{x}, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi})$, $i = 1, 2, \cdots, m$, respectively, where $\boldsymbol{\xi}$ is an uncertain vector.

Let the feasible set of control vector $\boldsymbol{x}$ of the leader be defined by the chance constraint

$$\mathcal{M}\{G(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha \tag{3.55}$$

where G is a constraint function, and α is a predetermined confidence level. Then for each decision $\boldsymbol{x}$ chosen by the leader, the feasibility of control vectors $\boldsymbol{y}_i$ of the ith followers should be dependent on not only $\boldsymbol{x}$ but also $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{i-1}, \boldsymbol{y}_{i+1}, \cdots, \boldsymbol{y}_m$, and generally represented by the chance constraints,

$$\mathcal{M}\{g_i(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi}) \le 0\} \ge \alpha_i \tag{3.56}$$

where g_i are constraint functions, and α_i are predetermined confidence levels, $i = 1, 2, \cdots, m$, respectively.

Assume that the leader first chooses his control vector $\boldsymbol{x}$, and the followers determine their control array $(\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m)$ after that. In order to minimize the expected objective of the leader, Liu-Yao [96] proposed the following uncertain multilevel programming,

$$\left\{\begin{array}{l} \min\limits_{\boldsymbol{x}} E[F(\boldsymbol{x}, \boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{G(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha \\ \qquad (\boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*) \text{ solves problems } (i = 1, 2, \cdots, m) \\ \qquad \left\{\begin{array}{l} \min\limits_{\boldsymbol{y_i}} E[f_i(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_i(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi}) \le 0\} \ge \alpha_i. \end{array}\right. \end{array}\right. \tag{3.57}$$

Definition 3.4 *Let* $\boldsymbol{x}$ *be a feasible control vector of the leader. A Nash equilibrium of followers is the feasible array* $(\boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*)$ *with respect to* $\boldsymbol{x}$ *if*

$$\begin{array}{l} E[f_i(\boldsymbol{x}, \boldsymbol{y}_1^*, \cdots, \boldsymbol{y}_{i-1}^*, \boldsymbol{y}_i, \boldsymbol{y}_{i+1}^*, \cdots, \boldsymbol{y}_m^*, \boldsymbol{\xi})] \\ \qquad \ge E[f_i(\boldsymbol{x}, \boldsymbol{y}_1^*, \cdots, \boldsymbol{y}_{i-1}^*, \boldsymbol{y}_i^*, \boldsymbol{y}_{i+1}^*, \cdots, \boldsymbol{y}_m^*, \boldsymbol{\xi})] \end{array} \tag{3.58}$$

for any feasible array $(\boldsymbol{y}_1^*, \cdots, \boldsymbol{y}_{i-1}^*, \boldsymbol{y}_i, \boldsymbol{y}_{i+1}^*, \cdots, \boldsymbol{y}_m^*)$ *and* $i = 1, 2, \cdots, m$.

Definition 3.5 *Suppose that* $\boldsymbol{x}^*$ *is a feasible control vector of the leader and* $(\boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*)$ *is a Nash equilibrium of followers with respect to* $\boldsymbol{x}^*$. *We call the array* $(\boldsymbol{x}^*, \boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*)$ *a Stackelberg-Nash equilibrium to the uncertain multilevel programming (3.57) if*

$$E[F(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}_1, \overline{\boldsymbol{y}}_2, \cdots, \overline{\boldsymbol{y}}_m, \boldsymbol{\xi})] \ge E[F(\boldsymbol{x}^*, \boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*, \boldsymbol{\xi})] \tag{3.59}$$

for any feasible control vector $\overline{\boldsymbol{x}}$ *and the Nash equilibrium* $(\overline{\boldsymbol{y}}_1, \overline{\boldsymbol{y}}_2, \cdots, \overline{\boldsymbol{y}}_m)$ *with respect to* $\overline{\boldsymbol{x}}$.

3.9 Bibliographic Notes

Uncertain programming was founded by Liu [78] in 2009 and was applied to machine scheduling problem, vehicle routing problem and project scheduling problem by Liu [83] in 2010.

As extensions of uncertain programming theory, Liu-Chen [95] developed an uncertain multiobjective programming and an uncertain goal programming. In addition, Liu-Yao [96] suggested an uncertain multilevel programming for modeling decentralized decision systems with uncertain factors.

After that, the uncertain programming has obtained fruitful results in both theory and practice. For exploring more books and papers, the interested reader may visit the website at http://orsc.edu.cn/online.

Chapter 4

Uncertain Risk Analysis

The term *risk* has been used in different ways in literature. Here the risk is defined as the "accidental loss" plus "uncertain measure of such loss". Uncertain risk analysis is a tool to quantify risk via uncertainty theory. One main feature of this topic is to model events that almost never occur. This chapter will introduce a definition of risk index and provide some useful formulas for calculating risk index. This chapter will also discuss structural risk analysis and investment risk analysis in uncertain environments.

4.1 Loss Function

A system usually contains some factors $\xi_1, \xi_2, \cdots, \xi_n$ that may be understood as lifetime, strength, demand, production rate, cost, profit, and resource. Generally speaking, some specified loss is dependent on those factors. Although loss is a problem-dependent concept, usually such a loss may be represented by a loss function.

Definition 4.1 *Consider a system with factors $\xi_1, \xi_2, \cdots, \xi_n$. A function f is called a loss function if some specified loss occurs if and only if*

$$f(\xi_1, \xi_2, \cdots, \xi_n) > 0. \tag{4.1}$$

Example 4.1: Consider a series system in which there are n elements whose lifetimes are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$. Such a system works whenever all elements work. Thus the system lifetime is

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n. \tag{4.2}$$

If the loss is understood as the case that the system fails before the time T, then we have a loss function

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n. \tag{4.3}$$

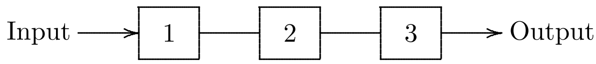


Figure 4.1: A Series System

Hence the system fails if and only if $f(\xi_1, \xi_2, \cdots, \xi_n) > 0$.

Example 4.2: Consider a parallel system in which there are n elements whose lifetimes are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$. Such a system works whenever at least one element works. Thus the system lifetime is

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n. \tag{4.4}$$

If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n. \tag{4.5}$$

Hence the system fails if and only if $f(\xi_1, \xi_2, \cdots, \xi_n) > 0$.

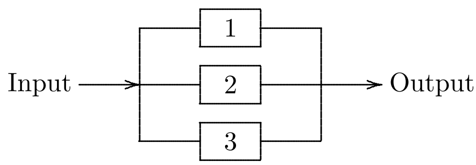


Figure 4.2: A Parallel System

Example 4.3: Consider a k-out-of-n system in which there are n elements whose lifetimes are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$. Such a system works whenever at least k of n elements work. Thus the system lifetime is

$$\xi = k\text{-max}\,[\xi_1, \xi_2, \cdots, \xi_n]. \tag{4.6}$$

If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - k\text{-max}\,[\xi_1, \xi_2, \cdots, \xi_n]. \tag{4.7}$$

Hence the system fails if and only if $f(\xi_1, \xi_2, \cdots, \xi_n) > 0$. Note that a series system is an n-out-of-n system, and a parallel system is a 1-out-of-n system.

Example 4.4: Consider a standby system in which there are n redundant elements whose lifetimes are $\xi_1, \xi_2, \cdots, \xi_n$. For this system, only one element is active, and one of the redundant elements begins to work only when the active element fails. Thus the system lifetime is

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n. \tag{4.8}$$

If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - (\xi_1 + \xi_2 + \cdots + \xi_n). \tag{4.9}$$

Hence the system fails if and only if $f(\xi_1, \xi_2, \cdots, \xi_n) > 0$.

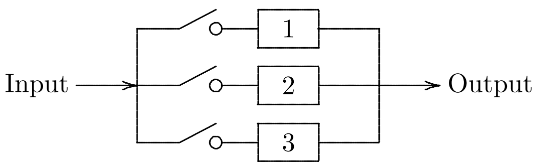


Figure 4.3: A Standby System

4.2 Risk Index

In practice, the factors $\xi_1, \xi_2, \cdots, \xi_n$ of a system are usually uncertain variables rather than known constants. Thus the risk index is defined as the uncertain measure that some specified loss occurs.

Definition 4.2 *(Liu [82]) Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$ and has a loss function f. Then the risk index is the uncertain measure that the system is loss-positive, i.e.,*

$$\mathit{Risk} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{4.10}$$

Theorem 4.1 *Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$, and has a loss function f. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ has an uncertainty distribution Φ, then the risk index is*

$$\mathit{Risk} = 1 - \Phi(0). \tag{4.11}$$

Proof: It follows from the definition of risk index and the duality axiom that

$$\begin{aligned}\mathit{Risk} &= \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}\\ &= 1 - \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \le 0\}\\ &= 1 - \Phi(0).\end{aligned}$$

The theorem is proved.

Theorem 4.2 *(Liu [82], Risk Index Theorem) Assume a system contains independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss function $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with* *respect to* $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$*, then the risk index is just the root* α *of the equation*

$$f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) = 0. \tag{4.12}$$

Proof: It follows from Theorem 2.14 that $f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Since $Risk = 1 - \Phi(0)$, it is the solution α of the equation $\Phi^{-1}(1-\alpha) = 0$. The theorem is thus proved.

Remark 4.1: Since $f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))$ is a strictly decreasing function with respect to α, its root α may be estimated by the bisection method.

Remark 4.2: Keep in mind that sometimes the equation (4.12) may not have a root. In this case, if

$$f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) < 0 \tag{4.13}$$

for all α, then we set the root $\alpha = 0$; and if

$$f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) > 0 \tag{4.14}$$

for all α, then we set the root $\alpha = 1$.

4.3 Series System

Consider a series system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{4.15}$$

and the risk index is

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{4.16}$$

Since f is a strictly decreasing function with respect to $\xi_1, \xi_2, \cdots, \xi_n$, the risk index theorem says that the risk index is just the root α of the equation

$$\Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha) \wedge \cdots \wedge \Phi_n^{-1}(\alpha) = T. \tag{4.17}$$

It is easy to verify that

$$Risk = \Phi_1(T) \vee \Phi_2(T) \vee \cdots \vee \Phi_n(T). \tag{4.18}$$

4.4 Parallel System

Consider a parallel system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{4.19}$$

and the risk index is

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{4.20}$$

Since f is a strictly decreasing function with respect to $\xi_1, \xi_2, \cdots, \xi_n$, the risk index theorem says that the risk index is just the root α of the equation

$$\Phi_1^{-1}(\alpha) \vee \Phi_2^{-1}(\alpha) \vee \cdots \vee \Phi_n^{-1}(\alpha) = T. \tag{4.21}$$

It is easy to verify that

$$Risk = \Phi_1(T) \wedge \Phi_2(T) \wedge \cdots \wedge \Phi_n(T). \tag{4.22}$$

4.5 k-out-of-n System

Consider a k-out-of-n system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - k\text{-max}\,[\xi_1, \xi_2, \cdots, \xi_n] \tag{4.23}$$

and the risk index is

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{4.24}$$

Since f is a strictly decreasing function with respect to $\xi_1, \xi_2, \cdots, \xi_n$, the risk index theorem says that the risk index is just the root α of the equation

$$k\text{-max}\,[\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)] = T. \tag{4.25}$$

It is easy to verify that

$$Risk = k\text{-min}\,[\Phi_1(T), \Phi_2(T), \cdots, \Phi_n(T)]. \tag{4.26}$$

Note that a series system is essentially an n-out-of-n system. In this case, the risk index formula (4.26) becomes (4.18). In addition, a parallel system is essentially a 1-out-of-n system. In this case, the risk index formula (4.26) becomes (4.22).

4.6 Standby System

Consider a standby system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - (\xi_1 + \xi_2 + \cdots + \xi_n) \tag{4.27}$$

and the risk index is

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{4.28}$$

Since f is a strictly decreasing function with respect to $\xi_1, \xi_2, \cdots, \xi_n$, the risk index theorem says that the risk index is just the root α of the equation

$$\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha) = T. \tag{4.29}$$

4.7 Structural Risk Analysis

Uncertain structural risk analysis was first investigated by Liu [94]. Consider a structural system in which the strengths and loads are assumed to be uncertain variables. We will suppose that a structural system fails whenever for each rod, the load variable exceeds its strength variable. If the structural risk index is defined as the uncertain measure that the structural system fails, then

$$Risk = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i < \eta_i)\right\} \tag{4.30}$$

where $\xi_1, \xi_2, \cdots, \xi_n$ are strength variables, and $\eta_1, \eta_2, \cdots, \eta_n$ are load variables of the n rods.

Example 4.5: (The Simplest Case) Assume there is only a single strength variable ξ and a single load variable η with regular uncertainty distributions Φ and Ψ, respectively. In this case, the structural risk index is

$$Risk = \mathcal{M}\{\xi < \eta\}.$$

It follows from the risk index theorem that the risk index is just the root α of the equation

$$\Phi^{-1}(\alpha) = \Psi^{-1}(1 - \alpha). \tag{4.31}$$

Especially, if the strength variable ξ has a normal uncertainty distribution $\mathcal{N}(e_s, \sigma_s)$ and the load variable η has a normal uncertainty distribution $\mathcal{N}(e_l, \sigma_l)$, then the structural risk index is

$$Risk = \left(1 + \exp\left(\frac{\pi(e_s - e_l)}{\sqrt{3}(\sigma_s + \sigma_l)}\right)\right)^{-1}. \tag{4.32}$$

Example 4.6: (Constant Loads) Assume the uncertain strength variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent and have continuous uncertainty distributions Φ_1, $\Phi_2, \cdots, \Phi_n$, respectively. In many cases, the load variables $\eta_1, \eta_2, \cdots, \eta_n$ degenerate to crisp values $c_1, c_2, \cdots, c_n$ (for example, weight limits allowed by the legislation), respectively. In this case, it follows from (4.30) and independence that the structural risk index is

$$Risk = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i < c_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i < c_i\}.$$

That is,

$$Risk = \Phi_1(c_1) \vee \Phi_2(c_2) \vee \cdots \vee \Phi_n(c_n). \tag{4.33}$$

Example 4.7: (Independent Load Variables) Assume the uncertain strength variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent and have regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Also assume the uncertain load variables $\eta_1, \eta_2, \cdots, \eta_n$ are independent and have regular uncertainty distributions $\Psi_1, \Psi_2, \cdots, \Psi_n$, respectively. In this case, it follows from (4.30) and independence that the structural risk index is

$$Risk = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i < \eta_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i < \eta_i\}.$$

That is,

$$Risk = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n \tag{4.34}$$

where α_i are the roots of the equations

$$\Phi_i^{-1}(\alpha) = \Psi_i^{-1}(1-\alpha) \tag{4.35}$$

for $i = 1, 2, \cdots, n$, respectively.

However, generally speaking, the load variables $\eta_1, \eta_2, \cdots, \eta_n$ are neither constants nor independent. For examples, the load variables $\eta_1, \eta_2, \cdots, \eta_n$ may be functions of independent uncertain variables $\tau_1, \tau_2, \cdots, \tau_m$. In this case, the formula (4.34) is no longer valid. Thus we have to deal with those structural systems case by case.

Example 4.8: (Series System) Consider a structural system shown in Figure 4.4 that consists of n rods in series and an object. Assume that the strength variables of the n rods are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. We also assume that the gravity of the object is an uncertain variable η with regular uncertainty distribution Ψ. For each i $(1 \le i \le n)$, the load variable of the rod i is just the gravity η of the object. Thus the structural system fails whenever the load variable η exceeds at least one of the strength variables $\xi_1, \xi_2, \cdots, \xi_n$. Hence the structural risk index is

$$Risk = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i < \eta)\right\} = \mathcal{M}\{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n < \eta\}.$$

Define the loss function as

$$f(\xi_1, \xi_2, \cdots, \xi_n, \eta) = \eta - \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n.$$

Then

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n, \eta) > 0\}.$$

Since the loss function f is strictly increasing with respect to η and strictly decreasing with respect to $\xi_1, \xi_2, \cdots, \xi_n$, it follows from the risk index theorem that the risk index is just the root α of the equation

$$\Psi^{-1}(1-\alpha) - \Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha) \wedge \cdots \wedge \Phi_n^{-1}(\alpha) = 0. \tag{4.36}$$

Or equivalently, let α_i be the roots of the equations

$$\Psi^{-1}(1-\alpha) = \Phi_i^{-1}(\alpha) \tag{4.37}$$

for $i = 1, 2, \cdots, n$, respectively. Then the structural risk index is

$$Risk = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n. \tag{4.38}$$

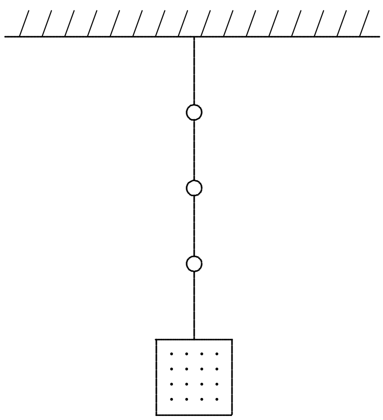


Figure 4.4: A Structural System with n Rods and an Object

Example 4.9: Consider a structural system shown in Figure 4.5 that consists of 2 rods and an object. Assume that the strength variables of the left and right rods are uncertain variables ξ_1 and ξ_2 with uncertainty distributions Φ_1 and Φ_2, respectively. We also assume that the gravity of the object is an uncertain variable η with regular uncertainty distribution Ψ. In this case, the load variables of left and right rods are respectively equal to

$$\frac{\eta\sin\theta_2}{\sin(\theta_1+\theta_2)}, \quad \frac{\eta\sin\theta_1}{\sin(\theta_1+\theta_2)}.$$

Thus the structural system fails whenever for any one rod, the load variable exceeds its strength variable. Hence the structural risk index is

$$\begin{aligned}
Risk &= \mathcal{M}\left\{\left(\xi_1 < \frac{\eta\sin\theta_2}{\sin(\theta_1+\theta_2)}\right) \cup \left(\xi_2 < \frac{\eta\sin\theta_1}{\sin(\theta_1+\theta_2)}\right)\right\} \\
&= \mathcal{M}\left\{\left(\frac{\xi_1}{\sin\theta_2} < \frac{\eta}{\sin(\theta_1+\theta_2)}\right) \cup \left(\frac{\xi_2}{\sin\theta_1} < \frac{\eta}{\sin(\theta_1+\theta_2)}\right)\right\} \\
&= \mathcal{M}\left\{\frac{\xi_1}{\sin\theta_2} \wedge \frac{\xi_2}{\sin\theta_1} < \frac{\eta}{\sin(\theta_1+\theta_2)}\right\}
\end{aligned}$$

Define the loss function as

$$f(\xi_1,\xi_2,\eta) = \frac{\eta}{\sin(\theta_1+\theta_2)} - \frac{\xi_1}{\sin\theta_2} \wedge \frac{\xi_2}{\sin\theta_1}.$$

Then

$$Risk = \mathcal{M}\{f(\xi_1,\xi_2,\eta) > 0\}.$$

Since the loss function f is strictly increasing with respect to η and strictly decreasing with respect to ξ_1, ξ_2, it follows from the risk index theorem that the risk index is just the root α of the equation

$$\frac{\Psi^{-1}(1-\alpha)}{\sin(\theta_1+\theta_2)} - \frac{\Phi_1^{-1}(\alpha)}{\sin\theta_2} \wedge \frac{\Phi_2^{-1}(\alpha)}{\sin\theta_1} = 0. \tag{4.39}$$

Or equivalently, let α_1 be the root of the equation

$$\frac{\Psi^{-1}(1-\alpha)}{\sin(\theta_1+\theta_2)} = \frac{\Phi_1^{-1}(\alpha)}{\sin\theta_2} \tag{4.40}$$

and let α_2 be the root of the equation

$$\frac{\Psi^{-1}(1-\alpha)}{\sin(\theta_1+\theta_2)} = \frac{\Phi_2^{-1}(\alpha)}{\sin\theta_1}. \tag{4.41}$$

Then the structural risk index is

$$Risk = \alpha_1 \vee \alpha_2. \tag{4.42}$$

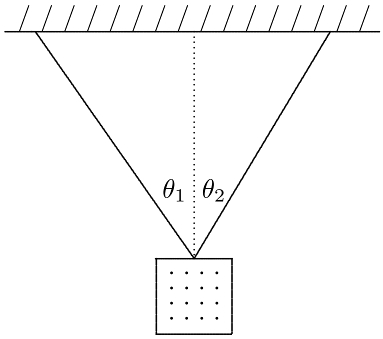


Figure 4.5: A Structural System with 2 Rods and an Object

4.8 Investment Risk Analysis

Uncertain investment risk analysis was first studied by Liu [94]. Assume that an investor has n projects whose returns are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$. If the loss is understood as the case that total return $\xi_1 + \xi_2 + \cdots + \xi_n$ is below a predetermined value c (e.g., the interest rate), then the investment risk index is

$$Risk = \mathcal{M}\{\xi_1 + \xi_2 + \cdots + \xi_n < c\}. \tag{4.43}$$

If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the investment risk index is just the root α of the equation

$$\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha) = c. \tag{4.44}$$

4.9 Value-at-Risk

As a substitute of risk index (4.10), a concept of value-at-risk is given by the following definition.

Definition 4.3 *(Peng [119]) Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$ and has a loss function f. Then the value-at-risk is defined as*

$$\mathrm{VaR}(\alpha) = \sup\{x \mid \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \ge x\} \ge \alpha\}. \tag{4.45}$$

Note that $\mathrm{VaR}(\alpha)$ represents the maximum possible loss when α percent of the right tail distribution is ignored. In other words, the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ will exceed $\mathrm{VaR}(\alpha)$ with uncertain measure α. See Figure 4.6. If the uncertainty distribution $\Phi(x)$ of $f(\xi_1, \xi_2, \cdots, \xi_n)$ is continuous, then

$$\mathrm{VaR}(\alpha) = \sup\{x \mid \Phi(x) \le 1 - \alpha\}. \tag{4.46}$$

If its inverse uncertainty distribution $\Phi^{-1}(\alpha)$ exists, then

$$\text{VaR}(\alpha) = \Phi^{-1}(1-\alpha). \tag{4.47}$$

It is also easy to show that $\text{VaR}(\alpha)$ is a monotone decreasing function with respect to α.

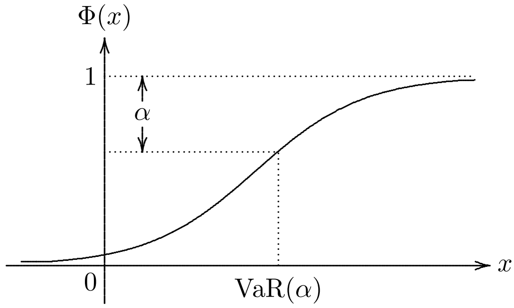


Figure 4.6: Value-at-Risk

Theorem 4.3 *(Peng [119], Value-at-Risk Theorem) Assume a system contains independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss function $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then*

$$\text{VaR}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)). \tag{4.48}$$

Proof: It follows from the operational law of uncertain variables that the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

The theorem follows from (4.47) immediately.

4.10 Expected Loss

Liu-Ralescu [111] proposed a concept of expected loss that is the expected value of the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ given $f(\xi_1, \xi_2, \cdots, \xi_n) > 0$. A formal definition is given below.

Definition 4.4 *(Liu-Ralescu [111]) Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$ and has a loss function f. Then the expected loss is defined as*

$$L = \int_0^{+\infty} \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \ge x\}\mathrm{d}x. \tag{4.49}$$

If $\Phi(x)$ is the uncertainty distribution of the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$, then we immediately have

$$L = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x. \tag{4.50}$$

If its inverse uncertainty distribution $\Phi^{-1}(\alpha)$ exists, then the expected loss is

$$L = \int_0^1 \left(\Phi^{-1}(\alpha)\right)^+ \mathrm{d}\alpha. \tag{4.51}$$

Theorem 4.4 *(Liu-Ralescu [111], Expected Loss Theorem) Assume that a system contains independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss function $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then the expected loss is*

$$L = \int_0^1 f^+(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \tag{4.52}$$

Proof: It follows from the operational law of uncertain variables that the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

The theorem follows from (4.51) immediately.

4.11 Hazard Distribution

Suppose that ξ is the lifetime of some element. Here we assume it is an uncertain variable with a prior uncertainty distribution Φ. At some time t, it is observed that the element is working. What is the residual lifetime of the element? The following definition answers this question.

Definition 4.5 *(Liu [82]) Let ξ be a nonnegative uncertain variable representing lifetime of some element. If ξ has a prior uncertainty distribution Φ, then the hazard distribution at time t is*

$$\Phi(x|t) = \begin{cases} 0, & \text{if } \Phi(x) \le \Phi(t) \\ \dfrac{\Phi(x)}{1-\Phi(t)} \wedge 0.5, & \text{if } \Phi(t) < \Phi(x) \le (1+\Phi(t))/2 \\ \dfrac{\Phi(x)-\Phi(t)}{1-\Phi(t)}, & \text{if } (1+\Phi(t))/2 \le \Phi(x) \end{cases} \tag{4.53}$$

that is just the conditional uncertainty distribution of ξ given $\xi > t$.

The hazard distribution is essentially the posterior uncertainty distribution just after time t given that it is working at time t.

Exercise 4.1: Let ξ be a linear uncertain variable $\mathcal{L}(a,b)$, and t a real number with $a<t<b$. Show that the hazard distribution at time t is

$$\Phi(x|t)=\begin{cases} 0, & \text{if } x\le t\\ \dfrac{x-a}{b-t}\wedge 0.5, & \text{if } t<x\le (b+t)/2\\ \dfrac{x-t}{b-t}\wedge 1, & \text{if } (b+t)/2\le x.\end{cases}$$

Theorem 4.5 *(Liu [82], Conditional Risk Index Theorem) Assume that a system contains uncertain factors $\xi_1,\xi_2,\cdots,\xi_n$, and has a loss function f. Suppose $\xi_1,\xi_2,\cdots,\xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively, and $f(\xi_1,\xi_2,\cdots,\xi_n)$ is strictly increasing with respect to $\xi_1,\xi_2,\cdots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\cdots,\xi_n$. If it is observed that all elements are working at some time t, then the risk index is just the root α of the equation*

$$f(\Phi_1^{-1}(1-\alpha|t),\cdots,\Phi_m^{-1}(1-\alpha|t),\Phi_{m+1}^{-1}(\alpha|t),\cdots,\Phi_n^{-1}(\alpha|t))=0 \tag{4.54}$$

where $\Phi_i(x|t)$ are hazard distributions determined by

$$\Phi_i(x|t)=\begin{cases} 0, & \text{if } \Phi_i(x)\le\Phi_i(t)\\ \dfrac{\Phi_i(x)}{1-\Phi_i(t)}\wedge 0.5, & \text{if } \Phi_i(t)<\Phi_i(x)\le(1+\Phi_i(t))/2\\ \dfrac{\Phi_i(x)-\Phi_i(t)}{1-\Phi_i(t)}, & \text{if } (1+\Phi_i(t))/2\le\Phi_i(x)\end{cases} \tag{4.55}$$

for $i=1,2,\cdots,n$.

Proof: It follows from Definition 4.5 that each hazard distribution of element is determined by (4.55). Thus the conditional risk index is obtained by Theorem 4.2 immediately.

Exercise 4.2: State and prove conditional value-at-risk theorem and conditional expected loss theorem.

4.12 Bibliographic Notes

Uncertain risk analysis was proposed by Liu [82] in 2010 in which the risk index was defined as the uncertain measure that some specified loss occurs, and a risk index theorem was proved. This tool was also successfully applied by Liu [94] to structural risk analysis and investment risk analysis.

As a substitute of risk index, Peng [119] suggested the concept of value-at-risk that is the maximum possible loss when the right tail distribution is ignored. In addition, Liu-Ralescu [111] investigated the concept of expected loss that takes into account not only the uncertain measure of the loss but also its severity.

Chapter 5

Uncertain Reliability Analysis

Uncertain reliability analysis is a tool to deal with system reliability via uncertainty theory. This chapter will introduce a definition of reliability index and provide some useful formulas for calculating the reliability index.

5.1 Structure Function

Many real systems may be simplified to a Boolean system in which each element (including the system itself) has two states: working and failure. We denote the states of elements i by the Boolean variables

$$x_i = \begin{cases} 1, & \text{if element } i \text{ works} \\ 0, & \text{if element } i \text{ fails,} \end{cases} \tag{5.1}$$

$i = 1, 2, \cdots, n$, respectively. We also denote the state of the system by the Boolean variable

$$X = \begin{cases} 1, & \text{if the system works} \\ 0, & \text{if the system fails.} \end{cases} \tag{5.2}$$

Usually, the state of the system is completely determined by the states of its elements via the so-called structure function.

Definition 5.1 *Assume that X is a Boolean system containing elements $x_1, x_2, \cdots, x_n$. A Boolean function f is called a structure function of X if*

$$X = 1 \text{ if and only if } f(x_1, x_2, \cdots, x_n) = 1. \tag{5.3}$$

It is obvious that $X = 0$ if and only if $f(x_1, x_2, \cdots, x_n) = 0$ whenever f is indeed the structure function of the system.

Example 5.1: For a series system, the structure function is a mapping from $\{0,1\}^n$ to $\{0,1\}$, i.e.,

$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{5.4}$$

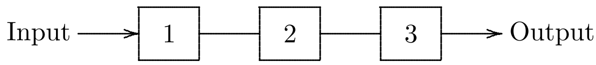


Figure 5.1: A Series System

Example 5.2: For a parallel system, the structure function is a mapping from $\{0,1\}^n$ to $\{0,1\}$, i.e.,

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n. \tag{5.5}$$

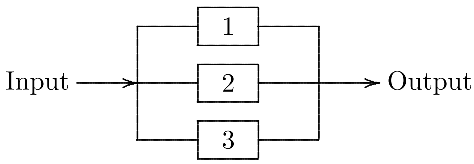


Figure 5.2: A Parallel System

Example 5.3: For a k-out-of-n system that works whenever at least k of the n elements work, the structure function is a mapping from $\{0,1\}^n$ to $\{0,1\}$, i.e.,

$$f(x_1, x_2, \cdots, x_n) = k\text{-max}\,[x_1, x_2, \cdots, x_n]. \tag{5.6}$$

Especially, when $k = 1$, it is a parallel system; when $k = n$, it is a series system.

5.2 Reliability Index

The element in a Boolean system is usually represented by a Boolean uncertain variable, i.e.,

$$\xi = \begin{cases} 1 \text{ with uncertain measure } a \\ 0 \text{ with uncertain measure } 1 - a. \end{cases} \tag{5.7}$$

In this case, we will say ξ is an uncertain element with reliability a. Reliability index is defined as the uncertain measure that the system is working.

Definition 5.2 *(Liu [82]) Assume a Boolean system has uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$ and a structure function f. Then the reliability index is the uncertain measure that the system is working, i.e.,*

$$\textit{Reliability} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) = 1\}. \tag{5.8}$$

Theorem 5.1 *(Liu [82], Reliability Index Theorem) Assume that a system contains uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$, and has a structure function f. If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain elements with reliabilities $a_1, a_2, \cdots, a_n$, respectively, then the reliability index is*

$$\textit{Reliability} = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{5.9}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \textit{if } x_i = 1 \\ 1 - a_i, & \textit{if } x_i = 0 \end{cases} \tag{5.10}$$

for $i = 1, 2, \cdots, n$, respectively.

Proof: Since $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables and f is a Boolean function, the equation (5.9) follows from Definition 5.2 and Theorem 2.21 immediately.

5.3 Series System

Consider a series system having independent uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$ with reliabilities $a_1, a_2, \cdots, a_n$, respectively. Note that the structure function is

$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{5.11}$$

It follows from the reliability index theorem that the reliability index is

$$\textit{Reliability} = \mathcal{M}\{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n = 1\} = a_1 \wedge a_2 \wedge \cdots \wedge a_n. \tag{5.12}$$

5.4 Parallel System

Consider a parallel system having independent uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$ with reliabilities $a_1, a_2, \cdots, a_n$, respectively. Note that the structure function is

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n. \tag{5.13}$$

It follows from the reliability index theorem that the reliability index is

$$\textit{Reliability} = \mathcal{M}\{\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n = 1\} = a_1 \vee a_2 \vee \cdots \vee a_n. \tag{5.14}$$

5.5 k-out-of-n System

Consider a k-out-of-n system having independent uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$ with reliabilities $a_1, a_2, \cdots, a_n$, respectively. Note that the structure function has a Boolean form,

$$f(x_1, x_2, \cdots, x_n) = k\text{-max}\,[x_1, x_2, \cdots, x_n]. \tag{5.15}$$

It follows from the reliability index theorem that the reliability index is the kth largest value of $a_1, a_2, \cdots, a_n$, i.e.,

$$\textit{Reliability} = k\text{-max}[a_1, a_2, \cdots, a_n]. \tag{5.16}$$

Note that a series system is essentially an n-out-of-n system. In this case, the reliability index formula (5.16) becomes (5.12). In addition, a parallel system is essentially a 1-out-of-n system. In this case, the reliability index formula (5.16) becomes (5.14).

5.6 General System

It is almost impossible to find an analytic formula of reliability risk for general systems. In this case, we have to employ a numerical method.

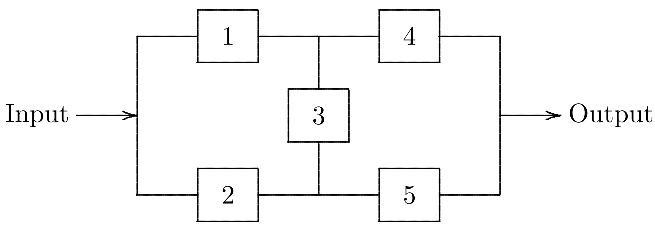


Figure 5.3: A Bridge System

Consider a bridge system shown in Figure 5.3 that consists of 5 independent uncertain elements whose states are denoted by $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5$. Assume each path works if and only if all elements on which are working and the system works if and only if there is a path of working elements. Then the structure function of the bridge system is

$$f(x_1, x_2, x_3, x_4, x_5) = (x_1 \wedge x_4) \vee (x_2 \wedge x_5) \vee (x_1 \wedge x_3 \wedge x_5) \vee (x_2 \wedge x_3 \wedge x_4).$$

The Boolean System Calculator, a function in the Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm), may yield the reliability index. Assume the 5 independent uncertain elements have reliabilities

$$0.91,\ 0.92,\ 0.93,\ 0.94,\ 0.95$$

in uncertain measure. A run of Boolean System Calculator shows that the reliability index is

$$\mathit{Reliability} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_5) = 1\} = 0.92$$

in uncertain measure.

5.7 Bibliographic Notes

Uncertain reliability analysis was proposed by Liu [82] in 2010 in which the reliability index was defined as the uncertain measure that the system is working, and a reliability index theorem was proved. After that, Zeng-Wen-Kang [192] and Gao-Yao [33] introduced some different reliability metrics for uncertain reliability systems.

Chapter 6

Uncertain Propositional Logic

Propositional logic, originated from the work of Aristotle (384-322 BC), is a branch of logic that studies the properties of complex propositions composed of simpler propositions and logical connectives. Note that the propositions considered in propositional logic are not arbitrary statements but are the ones that are either true or false and not both.

Uncertain propositional logic is a generalization of propositional logic in which every proposition is abstracted into a Boolean uncertain variable and the truth value is defined as the uncertain measure that the proposition is true. This chapter will deal with uncertain propositional logic, including uncertain proposition, truth value definition, and truth value theorem. This chapter will also introduce uncertain predicate logic.

6.1 Uncertain Proposition

Definition 6.1 *(Li-Liu [71]) An uncertain proposition is a statement whose truth value is quantified by an uncertain measure.*

That is, if we use X to express an uncertain proposition and use α to express its truth value in uncertain measure, then the uncertain proposition X is essentially a Boolean uncertain variable

$$X = \begin{cases} 1 \text{ with uncertain measure } \alpha \\ 0 \text{ with uncertain measure } 1-\alpha \end{cases} \tag{6.1}$$

where $X = 1$ means X is true and $X = 0$ means X is false.

Example 6.1: "Tom is tall with truth value 0.7" is an uncertain proposition, where "Tom is tall" is a statement, and its truth value is 0.7 in uncertain measure.

Example 6.2: "John is young with truth value 0.8" is an uncertain proposition, where "John is young" is a statement, and its truth value is 0.8 in uncertain measure.

Example 6.3: "Beijing is a big city with truth value 0.9" is an uncertain proposition, where "Beijing is a big city" is a statement, and its truth value is 0.9 in uncertain measure.

Connective Symbols

In addition to the proposition symbols X and Y, we also need the negation symbol $\neg$, conjunction symbol $\wedge$, disjunction symbol $\vee$, conditional symbol $\rightarrow$, and biconditional symbol $\leftrightarrow$. Note that

$$\neg X \text{ means "not } X\text{";} \tag{6.2}$$

$$X \wedge Y \text{ means "} X \text{ and } Y\text{";} \tag{6.3}$$

$$X \vee Y \text{ means "} X \text{ or } Y\text{";} \tag{6.4}$$

$$X \rightarrow Y = (\neg X) \vee Y \text{ means "if } X \text{ then } Y\text{",} \tag{6.5}$$

$$X \leftrightarrow Y = (X \rightarrow Y) \wedge (Y \rightarrow X) \text{ means "} X \text{ if and only if } Y\text{".} \tag{6.6}$$

Boolean Function of Uncertain Propositions

Assume $X_1, X_2, \cdots, X_n$ are uncertain propositions. Then their Boolean function

$$Z = f(X_1, X_2, \cdots, X_n) \tag{6.7}$$

is a Boolean uncertain variable. Thus Z is also an uncertain proposition provided that it makes sense. Usually, such a Boolean function is a finite sequence of uncertain propositions and connective symbols. For example,

$$Z = \neg X_1, \quad Z = X_1 \wedge (\neg X_2), \quad Z = X_1 \rightarrow X_2 \tag{6.8}$$

are all uncertain propositions.

Independence of Uncertain Propositions

Uncertain propositions are called *independent* if they are independent uncertain variables. Assume $X_1, X_2, \cdots, X_n$ are independent uncertain propositions. Then

$$f_1(X_1),\ f_2(X_2) \cdots, f_n(X_n) \tag{6.9}$$

are also independent uncertain propositions for any Boolean functions $f_1, f_2, \cdots, f_n$. For example, if $X_1, X_2, \cdots, X_5$ are independent uncertain propositions, then $\neg X_1, X_2 \vee X_3, X_4 \rightarrow X_5$ are also independent.

6.2 Truth Value

Truth value is a key concept in uncertain propositional logic, and is defined as the uncertain measure that the uncertain proposition is true.

Definition 6.2 *(Li-Liu [71]) Let X be an uncertain proposition. Then the truth value of X is defined as the uncertain measure that X is true, i.e.,*

$$T(X) = \mathcal{M}\{X = 1\}. \tag{6.10}$$

Example 6.4: Let X be an uncertain proposition with truth value α. Then

$$T(\neg X) = \mathcal{M}\{X = 0\} = 1 - \alpha. \tag{6.11}$$

Example 6.5: Let X and Y be two independent uncertain propositions with truth values α and β, respectively. Then

$$T(X \wedge Y) = \mathcal{M}\{X \wedge Y = 1\} = \mathcal{M}\{(X = 1) \cap (Y = 1)\} = \alpha \wedge \beta, \tag{6.12}$$

$$T(X \vee Y) = \mathcal{M}\{X \vee Y = 1\} = \mathcal{M}\{(X = 1) \cup (Y = 1)\} = \alpha \vee \beta, \tag{6.13}$$

$$T(X \to Y) = T(\neg X \vee Y) = (1 - \alpha) \vee \beta. \tag{6.14}$$

Theorem 6.1 *(Law of Excluded Middle) Let X be an uncertain proposition. Then $X \vee \neg X$ is a tautology, i.e.,*

$$T(X \vee \neg X) = 1. \tag{6.15}$$

Proof: It follows from the definition of truth value and the property of uncertain measure that

$$T(X \vee \neg X) = \mathcal{M}\{X \vee \neg X = 1\} = \mathcal{M}\{(X = 1) \cup (X = 0)\} = \mathcal{M}\{\Gamma\} = 1.$$

The theorem is proved.

Theorem 6.2 *(Law of Contradiction) Let X be an uncertain proposition. Then $X \wedge \neg X$ is a contradiction, i.e.,*

$$T(X \wedge \neg X) = 0. \tag{6.16}$$

Proof: It follows from the definition of truth value and the property of uncertain measure that

$$T(X \wedge \neg X) = \mathcal{M}\{X \wedge \neg X = 1\} = \mathcal{M}\{(X = 1) \cap (X = 0)\} = \mathcal{M}\{\emptyset\} = 0.$$

The theorem is proved.

Theorem 6.3 *(Law of Truth Conservation) Let X be an uncertain proposition. Then we have*

$$T(X) + T(\neg X) = 1. \tag{6.17}$$

Proof: It follows from the duality axiom of uncertain measure that

$$T(\neg X) = \mathcal{M}\{\neg X = 1\} = \mathcal{M}\{X = 0\} = 1 - \mathcal{M}\{X = 1\} = 1 - T(X).$$

The theorem is proved.

Theorem 6.4 *Let X be an uncertain proposition. Then $X \to X$ is a tautology, i.e.,*

$$T(X \to X) = 1. \tag{6.18}$$

Proof: It follows from the definition of conditional symbol and the law of excluded middle that

$$T(X \to X) = T(\neg X \vee X) = 1.$$

The theorem is proved.

Theorem 6.5 *Let X be an uncertain proposition. Then we have*

$$T(X \to \neg X) = 1 - T(X). \tag{6.19}$$

Proof: It follows from the definition of conditional symbol and the law of truth conservation that

$$T(X \to \neg X) = T(\neg X \vee \neg X) = T(\neg X) = 1 - T(X).$$

The theorem is proved.

Theorem 6.6 *(De Morgan's Law) For any uncertain propositions X and Y, we have*

$$T(\neg(X \wedge Y)) = T((\neg X) \vee (\neg Y)), \tag{6.20}$$

$$T(\neg(X \vee Y)) = T((\neg X) \wedge (\neg Y)). \tag{6.21}$$

Proof: It follows from the basic properties of uncertain measure that

$$\begin{aligned} T(\neg(X \wedge Y)) &= \mathcal{M}\{X \wedge Y = 0\} = \mathcal{M}\{(X = 0) \cup (Y = 0)\} \\ &= \mathcal{M}\{(\neg X) \vee (\neg Y) = 1\} = T((\neg X) \vee (\neg Y)) \end{aligned}$$

which proves the first equality. A similar way may verify the second equality.

Theorem 6.7 *(Law of Contraposition) For any uncertain propositions X and Y, we have*

$$T(X \to Y) = T(\neg Y \to \neg X). \tag{6.22}$$

Proof: It follows from the definition of conditional symbol and basic properties of uncertain measure that

$$\begin{aligned} T(X \to Y) &= \mathcal{M}\{(\neg X) \vee Y = 1\} = \mathcal{M}\{(X = 0) \cup (Y = 1)\} \\ &= \mathcal{M}\{Y \vee (\neg X) = 1\} = T(\neg Y \to \neg X). \end{aligned}$$

The theorem is proved.

6.3 Chen-Ralescu Theorem

An important contribution to uncertain propositional logic is the Chen-Ralescu theorem that provides a numerical method for calculating the truth values of uncertain propositions.

Theorem 6.8 *(Chen-Ralescu Theorem [7]) Assume that $X_1, X_2, \cdots, X_n$ are independent uncertain propositions with truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. Then for a Boolean function f, the uncertain proposition*

$$Z = f(X_1, X_2, \cdots, X_n). \tag{6.23}$$

has a truth value

$$T(Z) = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \textit{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{6.24}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} \alpha_i, & \textit{if } x_i = 1 \\ 1 - \alpha_i, & \textit{if } x_i = 0 \end{cases} \tag{6.25}$$

for $i = 1, 2, \cdots, n$, respectively.

Proof: Since $Z = 1$ if and only if $f(X_1, X_2, \cdots, X_n) = 1$, we immediately have

$$T(Z) = \mathcal{M}\{f(X_1, X_2, \cdots, X_n) = 1\}.$$

Thus the equation (6.24) follows from Theorem 2.21 immediately.

Example 6.6: Let X_1 and X_2 be independent uncertain propositions with truth values α_1 and α_2, respectively. Then

$$Z = X_1 \leftrightarrow X_2 \tag{6.26}$$

is an uncertain proposition. It is clear that $Z = f(X_1, X_2)$ if we define

$$f(1,1) = 1, \quad f(1,0) = 0, \quad f(0,1) = 0, \quad f(0,0) = 1.$$

At first, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = \max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge (1-\alpha_2)\},$$

$$\sup_{f(x_1,x_2)=0} \min_{1\le i\le 2} \nu_i(x_i) = \max\{(1-\alpha_1)\wedge\alpha_2, \alpha_1\wedge(1-\alpha_2)\}.$$

When $\alpha_1 \ge 0.5$ and $\alpha_2 \ge 0.5$, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = \alpha_1\wedge\alpha_2 \ge 0.5.$$

It follows from Chen-Ralescu theorem that

$$T(Z) = 1 - \sup_{f(x_1,x_2)=0} \min_{1\le i\le 2} \nu_i(x_i) = 1-(1-\alpha_1)\vee(1-\alpha_2) = \alpha_1\wedge\alpha_2.$$

When $\alpha_1 \ge 0.5$ and $\alpha_2 < 0.5$, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = (1-\alpha_1)\vee\alpha_2 \le 0.5.$$

It follows from Chen-Ralescu theorem that

$$T(Z) = \sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = (1-\alpha_1)\vee\alpha_2.$$

When $\alpha_1 < 0.5$ and $\alpha_2 \ge 0.5$, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = \alpha_1\vee(1-\alpha_2) \le 0.5.$$

It follows from Chen-Ralescu theorem that

$$T(Z) = \sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = \alpha_1\vee(1-\alpha_2).$$

When $\alpha_1 < 0.5$ and $\alpha_2 < 0.5$, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = (1-\alpha_1)\wedge(1-\alpha_2) > 0.5.$$

It follows from Chen-Ralescu theorem that

$$T(Z) = 1 - \sup_{f(x_1,x_2)=0} \min_{1\le i\le 2} \nu_i(x_i) = 1-\alpha_1\vee\alpha_2 = (1-\alpha_1)\wedge(1-\alpha_2).$$

Thus we have

$$T(Z) = \begin{cases} \alpha_1\wedge\alpha_2, & \text{if } \alpha_1\ge 0.5 \text{ and } \alpha_2\ge 0.5\\ (1-\alpha_1)\vee\alpha_2, & \text{if } \alpha_1\ge 0.5 \text{ and } \alpha_2< 0.5\\ \alpha_1\vee(1-\alpha_2), & \text{if } \alpha_1< 0.5 \text{ and } \alpha_2\ge 0.5\\ (1-\alpha_1)\wedge(1-\alpha_2), & \text{if } \alpha_1< 0.5 \text{ and } \alpha_2< 0.5. \end{cases} \tag{6.27}$$

Example 6.7: The independence condition in Theorem 6.8 cannot be removed. For example, take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\}=\mathcal{M}\{\gamma_2\}=0.5$. Then

$$X_1(\gamma) = \begin{cases} 0, & \text{if } \gamma=\gamma_1\\ 1, & \text{if } \gamma=\gamma_2 \end{cases} \tag{6.28}$$

is an uncertain proposition with truth value

$$T(X_1) = 0.5, \tag{6.29}$$

and

$$X_2(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \end{cases} \tag{6.30}$$

is also an uncertain proposition with truth value

$$T(X_2) = 0.5. \tag{6.31}$$

Note that X_1 and X_2 are not independent, and $X_1 \vee X_2 \equiv 1$ from which we obtain

$$T(X_1 \vee X_2) = 1. \tag{6.32}$$

However, by using (6.24), we get

$$T(X_1 \vee X_2) = 0.5. \tag{6.33}$$

Thus the independence condition cannot be removed.

Exercise 6.1: Let $X_1, X_2, \cdots, X_n$ be independent uncertain propositions with truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. Then

$$Z = X_1 \wedge X_2 \wedge \cdots \wedge X_n \tag{6.34}$$

is an uncertain proposition. Show that the truth value of Z is

$$T(Z) = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n. \tag{6.35}$$

Exercise 6.2: Let $X_1, X_2, \cdots, X_n$ be independent uncertain propositions with truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. Then

$$Z = X_1 \vee X_2 \vee \cdots \vee X_n \tag{6.36}$$

is an uncertain proposition. Show that the truth value of Z is

$$T(Z) = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n. \tag{6.37}$$

Exercise 6.3: Let X_1 and X_2 be independent uncertain propositions with truth values α_1 and α_2, respectively. (i) What is the truth value of $(X_1 \wedge X_2) \to X_2$? (ii) What is the truth value of $(X_1 \vee X_2) \to X_2$? (iii) What is the truth value of $X_1 \to (X_1 \wedge X_2)$? (iv) What is the truth value of $X_1 \to (X_1 \vee X_2)$?

Exercise 6.4: Let X_1, X_2, X_3 be independent uncertain propositions with truth values $\alpha_1, \alpha_2, \alpha_3$, respectively. What is the truth value of

$$X_1 \wedge (X_1 \vee X_2) \wedge (X_1 \vee X_2 \vee X_3)? \tag{6.38}$$

6.4 Boolean System Calculator

Boolean System Calculator is a software that may compute the truth value of uncertain proposition. This software may be downloaded from the website at http://orsc.edu.cn/liu/resources.htm. For example, assume X_1, X_2, X_3, X_4, X_5 are independent uncertain propositions with truth values $0.1, 0.3, 0.5, 0.7, 0.9$, respectively. Consider an uncertain proposition,

$$Z = (X_1 \wedge X_2) \vee (X_2 \wedge X_3) \vee (X_3 \wedge X_4) \vee (X_4 \wedge X_5). \tag{6.39}$$

It is clear that the corresponding Boolean function of Z has the form

$$f(x_1, x_2, x_3, x_4, x_5) = \begin{cases} 1, & \text{if } x_1 + x_2 = 2 \\ 1, & \text{if } x_2 + x_3 = 2 \\ 1, & \text{if } x_3 + x_4 = 2 \\ 1, & \text{if } x_4 + x_5 = 2 \\ 0, & \text{otherwise.} \end{cases}$$

A run of Boolean System Calculator shows that the truth value of Z is 0.7 in uncertain measure.

6.5 Uncertain Predicate Logic

Consider the following propositions: "Beijing is a big city", and "Tianjin is a big city". Uncertain propositional logic treats them as unrelated propositions. However, uncertain predicate logic represents them by a predicate proposition $X(a)$. If a represents Beijing, then

$$X(a) = \text{“Beijing is a big city”}. \tag{6.40}$$

If a represents Tianjin, then

$$X(a) = \text{“Tianjin is a big city”}. \tag{6.41}$$

Definition 6.3 *(Zhang-Li [197]) Uncertain predicate proposition is a sequence of uncertain propositions indexed by one or more parameters.*

In order to deal with uncertain predicate propositions, we need a universal quantifier $\forall$ and an existential quantifier $\exists$. If $X(a)$ is an uncertain predicate proposition defined by (6.40) and (6.41), then

$$(\forall a)X(a) = \text{“Both Beijing and Tianjin are big cities”}, \tag{6.42}$$

$$(\exists a)X(a) = \text{“At least one of Beijing and Tianjin is a big city”}. \tag{6.43}$$

Theorem 6.9 *(Zhang-Li [197], Law of Excluded Middle) Let $X(a)$ be an uncertain predicate proposition. Then*

$$T((\forall a)X(a) \vee (\exists a)\neg X(a)) = 1. \tag{6.44}$$

Proof: Since $\neg(\forall a)X(a) = (\exists a)\neg X(a)$, it follows from the definition of truth value and the property of uncertain measure that

$$T((\forall a)X(a) \vee (\exists a)\neg X(a)) = \mathcal{M}\{((\forall a)X(a) = 1) \cup ((\forall a)X(a) = 0)\} = 1.$$

The theorem is proved.

Theorem 6.10 *(Zhang-Li [197], Law of Contradiction) Let $X(a)$ be an uncertain predicate proposition. Then*

$$T((\forall a)X(a) \wedge (\exists a)\neg X(a)) = 0. \tag{6.45}$$

Proof: Since $\neg(\forall a)X(a) = (\exists a)\neg X(a)$, it follows from the definition of truth value and the property of uncertain measure that

$$T((\forall a)X(a) \wedge (\exists a)\neg X(a)) = \mathcal{M}\{((\forall a)X(a) = 1) \cap ((\forall a)X(a) = 0)\} = 0.$$

The theorem is proved.

Theorem 6.11 *(Zhang-Li [197], Law of Truth Conservation) Let $X(a)$ be an uncertain predicate proposition. Then*

$$T((\forall a)X(a)) + T((\exists a)\neg X(a)) = 1. \tag{6.46}$$

Proof: Since $\neg(\forall a)X(a) = (\exists a)\neg X(a)$, it follows from the definition of truth value and the property of uncertain measure that

$$T((\exists a)\neg X(a)) = 1 - \mathcal{M}\{(\forall a)X(a) = 1\} = 1 - T((\forall a)X(a)).$$

The theorem is proved.

Theorem 6.12 *(Zhang-Li [197]) Let $X(a)$ be an uncertain predicate proposition. Then for any given b, we have*

$$T((\forall a)X(a) \to X(b)) = 1. \tag{6.47}$$

Proof: The argument breaks into two cases. Case 1: If $X(b) = 0$, then $(\forall a)X(a) = 0$ and $\neg(\forall a)X(a) = 1$. Thus

$$(\forall a)X(a) \to X(b) = \neg(\forall a)X(a) \vee X(b) = 1.$$

Case II: If $X(b) = 1$, then we immediately have

$$(\forall a)X(a) \to X(b) = \neg(\forall a)X(a) \vee X(b) = 1.$$

Thus we always have (6.47). The theorem is proved.

Theorem 6.13 *(Zhang-Li [197]) Let $X(a)$ be an uncertain predicate proposition. Then for any given b, we have*

$$T(X(b) \to (\exists a)X(a)) = 1. \tag{6.48}$$

Proof: The argument breaks into two cases. Case 1: If $X(b) = 0$, then $\neg X(b) = 1$ and

$$X(b) \to (\forall a)X(a) = \neg X(b) \vee (\exists a)X(a) = 1.$$

Case II: If $X(b) = 1$, then $(\exists a)X(a) = 1$ and

$$X(b) \to (\exists a)X(a) = \neg X(b) \vee (\exists a)X(a) = 1.$$

Thus we always have (6.48). The theorem is proved.

Theorem 6.14 *(Zhang-Li [197]) Let $X(a)$ be an uncertain predicate proposition. Then*

$$T((\forall a)X(a) \to (\exists a)X(a)) = 1. \tag{6.49}$$

Proof: The argument breaks into two cases. Case 1: If $(\forall a)X(a) = 0$, then $\neg(\forall a)X(a) = 1$ and

$$(\forall a)X(a) \to (\exists a)X(a) = \neg(\forall a)X(a) \vee (\exists a)X(a) = 1.$$

Case II: If $(\forall a)X(a) = 1$, then $(\exists a)X(a) = 1$ and

$$(\forall a)X(a) \to (\exists a)X(a) = \neg(\forall a)X(a) \vee (\exists a)X(a) = 1.$$

Thus we always have (6.49). The theorem is proved.

Theorem 6.15 *(Zhang-Li [197]) Let $X(a)$ be an uncertain predicate proposition such that $\{X(a)|a \in A\}$ is a class of independent uncertain propositions. Then*

$$T((\forall a)X(a)) = \inf_{a \in A} T(X(a)), \tag{6.50}$$

$$T((\exists a)X(a)) = \sup_{a \in A} T(X(a)). \tag{6.51}$$

Proof: For each uncertain predicate proposition $X(a)$, by the meaning of universal quantifier, we obtain

$$T((\forall a)X(a)) = \mathcal{M}\{(\forall a)X(a) = 1\} = \mathcal{M}\left\{\bigcap_{a \in A}(X(a) = 1)\right\}.$$

Since $\{X(a)|a \in A\}$ is a class of independent uncertain propositions, we get

$$T((\forall a)X(a)) = \inf_{a \in A} \mathcal{M}\{X(a) = 1\} = \inf_{a \in A} T(X(a)).$$

The first equation is verified. Similarly, by the meaning of existential quantifier, we obtain

$$T((\exists a)X(a)) = \mathcal{M}\{(\exists a)X(a) = 1\} = \mathcal{M}\left\{\bigcup_{a \in A}(X(a) = 1)\right\}.$$

Since $\{X(a)|a \in A\}$ is a class of independent uncertain propositions, we get

$$T((\exists a)X(a)) = \sup_{a \in A}\mathcal{M}\{X(a) = 1\} = \sup_{a \in A} T(X(a)).$$

The second equation is proved.

Theorem 6.16 *(Zhang-Li [197]) Let $X(a, b)$ be an uncertain predicate proposition such that $\{X(a, b)|a \in A, b \in B\}$ is a class of independent uncertain propositions. Then*

$$T((\forall a)(\exists b)X(a, b)) = \inf_{a \in A}\sup_{b \in B} T(X(a, b)), \tag{6.52}$$

$$T((\exists a)(\forall b)X(a, b)) = \sup_{a \in A}\inf_{b \in B} T(X(a, b)). \tag{6.53}$$

Proof: Since $\{X(a, b)|a \in A, b \in B\}$ is a class of independent uncertain propositions, both $\{(\exists b)X(a, b)|a \in A\}$ and $\{(\forall b)X(a, b)|a \in A\}$ are two classes of independent uncertain propositions. It follows from Theorem 6.15 that

$$T((\forall a)(\exists b)X(a, b)) = \inf_{a \in A} T((\exists b)X(a, b)) = \inf_{a \in A}\sup_{b \in B} T(X(a, b)),$$

$$T((\exists a)(\forall b)X(a, b)) = \sup_{a \in A} T((\forall b)X(a, b)) = \sup_{a \in A}\inf_{b \in B} T(X(a, b)).$$

The theorem is proved.

6.6 Bibliographic Notes

Uncertain propositional logic was designed by Li-Liu [71] in which every proposition is abstracted into a Boolean uncertain variable and the truth value is defined as the uncertain measure that the proposition is true. An important contribution is Chen-Ralescu theorem [7] that provides a numerical method for calculating the truth value of uncertain propositions.

Another topic is the uncertain predicate logic developed by Zhang-Li [197] in which an uncertain predicate proposition is defined as a sequence of uncertain propositions indexed by one or more parameters.

Chapter 7

Uncertain Entailment

Uncertain entailment is a methodology for calculating the truth value of an uncertain formula via the maximum uncertainty principle when the truth values of other uncertain formulas are given. In some sense, uncertain propositional logic and uncertain entailment are mutually inverse, the former attempts to compose a complex proposition from simpler ones, while the latter attempts to decompose a complex proposition into simpler ones.

This chapter will present an uncertain entailment model. In addition, uncertain modus ponens, uncertain modus tollens and uncertain hypothetical syllogism are deduced from the uncertain entailment model.

7.1 Uncertain Entailment Model

Assume $X_1, X_2, \cdots, X_n$ are independent uncertain propositions with *unknown* truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. Also assume that

$$Y_j = f_j(X_1, X_2, \cdots, X_n) \tag{7.1}$$

are uncertain propositions with *known* truth values c_j, $j = 1, 2, \cdots, m$, respectively. Now let

$$Z = f(X_1, X_2, \cdots, X_n) \tag{7.2}$$

be an additional uncertain proposition. What is the truth value of Z? This is just the uncertain entailment problem. In order to solve it, let us consider what values $\alpha_1, \alpha_2, \cdots, \alpha_n$ may take. The first constraint is

$$0 \le \alpha_i \le 1, \quad i = 1, 2, \cdots, n. \tag{7.3}$$

The second type of constraints is represented by

$$T(Y_j) = c_j \tag{7.4}$$

where $T(Y_j)$ are determined by $\alpha_1, \alpha_2, \cdots, \alpha_n$ via

$$T(Y_j) = \begin{cases} \sup\limits_{f_j(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if } \sup\limits_{f_j(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f_j(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if } \sup\limits_{f_j(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{7.5}$$

for $j = 1, 2, \cdots, m$ and

$$\nu_i(x_i) = \begin{cases} \alpha_i, & \text{if } x_i = 1 \\ 1 - \alpha_i, & \text{if } x_i = 0 \end{cases} \tag{7.6}$$

for $i = 1, 2, \cdots, n$. Please note that the additional uncertain proposition $Z = f(X_1, X_2, \cdots, X_n)$ has a truth value

$$T(Z) = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5. \end{cases} \tag{7.7}$$

Since the truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not uniquely determined, the truth value $T(Z)$ is not unique too. In this case, we have to use the maximum uncertainty principle to determine the truth value $T(Z)$. That is, $T(Z)$ should be assigned the value as close to 0.5 as possible. In other words, we should minimize the value $|T(Z) - 0.5|$ via choosing appreciate values of $\alpha_1, \alpha_2, \cdots, \alpha_n$. The uncertain entailment model is thus written by Liu [80] as follows,

$$\begin{cases} \min |T(Z) - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_i \le 1, \quad i = 1, 2, \cdots, n \\ \qquad T(Y_j) = c_j, \quad j = 1, 2, \cdots, m \end{cases} \tag{7.8}$$

where $T(Z), T(Y_j), j = 1, 2, \cdots, m$ are functions of unknown truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Example 7.1: Let A and B be independent uncertain propositions. It is known that

$$T(A \vee B) = a, \quad T(A \wedge B) = b. \tag{7.9}$$

What is the truth value of $A \to B$? Denote the truth values of A and B by α_1 and α_2, respectively, and write

$$Y_1 = A \vee B, \quad Y_2 = A \wedge B, \quad Z = A \to B.$$

It is clear that

$$T(Y_1) = \alpha_1 \vee \alpha_2 = a,$$
$$T(Y_2) = \alpha_1 \wedge \alpha_2 = b,$$
$$T(Z) = (1 - \alpha_1) \vee \alpha_2.$$

In this case, the uncertain entailment model (7.8) becomes

$$\begin{cases} \min |(1-\alpha_1) \vee \alpha_2 - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad \alpha_1 \vee \alpha_2 = a \\ \qquad \alpha_1 \wedge \alpha_2 = b. \end{cases} \tag{7.10}$$

When $a \ge b$, there are only two feasible solutions $(\alpha_1, \alpha_2) = (a, b)$ and $(\alpha_1, \alpha_2) = (b, a)$. If $a + b < 1$, the optimal solution produces

$$T(Z) = (1 - \alpha_1^*) \vee \alpha_2^* = 1 - a;$$

if $a + b = 1$, the optimal solution produces

$$T(Z) = (1 - \alpha_1^*) \vee \alpha_2^* = a \text{ or } b;$$

if $a + b > 1$, the optimal solution produces

$$T(Z) = (1 - \alpha_1^*) \vee \alpha_2^* = b.$$

When $a < b$, there is no feasible solution and the truth values are ill-assigned. In summary, from $T(A \vee B) = a$ and $T(A \wedge B) = b$ we entail

$$T(A \to B) = \begin{cases} 1 - a, & \text{if } a \ge b \text{ and } a + b < 1 \\ a \text{ or } b, & \text{if } a \ge b \text{ and } a + b = 1 \\ b, & \text{if } a \ge b \text{ and } a + b > 1 \\ \text{illness}, & \text{if } a < b. \end{cases} \tag{7.11}$$

Exercise 7.1: Let A, B, C be independent uncertain propositions. It is known that

$$T(A \to C) = a, \quad T(B \to C) = b, \quad T(A \vee B) = c. \tag{7.12}$$

What is the truth value of C?

Exercise 7.2: Let A, B, C, D be independent uncertain propositions. It is known that

$$T(A \to C) = a, \quad T(B \to D) = b, \quad T(A \vee B) = c. \tag{7.13}$$

What is the truth value of $C \vee D$?

Exercise 7.3: Let A, B, C be independent uncertain propositions. It is known that

$$T(A \vee B) = a, \quad T(\neg A \vee C) = b. \tag{7.14}$$

What is the truth value of $B \vee C$?

7.2 Uncertain Modus Ponens

Uncertain modus ponens was presented by Liu [80]. Let A and B be independent uncertain propositions. Assume A and $A \to B$ have truth values a and b, respectively. What is the truth value of B? Denote the truth values of A and B by α_1 and α_2, respectively, and write

$$Y_1 = A, \quad Y_2 = A \to B, \quad Z = B.$$

It is clear that

$$T(Y_1) = \alpha_1 = a,$$

$$T(Y_2) = (1 - \alpha_1) \vee \alpha_2 = b,$$

$$T(Z) = \alpha_2.$$

In this case, the uncertain entailment model (7.8) becomes

$$\begin{cases} \min |\alpha_2 - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad \alpha_1 = a \\ \qquad (1 - \alpha_1) \vee \alpha_2 = b. \end{cases} \tag{7.15}$$

When $a + b > 1$, there is a unique feasible solution and then the optimal solution is

$$\alpha_1^* = a, \quad \alpha_2^* = b.$$

Thus $T(B) = \alpha_2^* = b$. When $a + b = 1$, the feasible set is $\{a\} \times [0, b]$ and the optimal solution is

$$\alpha_1^* = a, \quad \alpha_2^* = 0.5 \wedge b.$$

Thus $T(B) = \alpha_2^* = 0.5 \wedge b$. When $a + b < 1$, there is no feasible solution and the truth values are ill-assigned. In summary, from

$$T(A) = a, \quad T(A \to B) = b \tag{7.16}$$

we entail

$$T(B) = \begin{cases} b, & \text{if } a + b > 1 \\ 0.5 \wedge b, & \text{if } a + b = 1 \\ \text{illness}, & \text{if } a + b < 1. \end{cases} \tag{7.17}$$

This result coincides with the classical modus ponens that if both A and $A \to B$ are true, then B is true.

7.3 Uncertain Modus Tollens

Uncertain modus tollens was presented by Liu [80]. Let A and B be independent uncertain propositions. Assume $A \to B$ and B have truth values a and b, respectively. What is the truth value of A? Denote the truth values of A and B by α_1 and α_2, respectively, and write

$$Y_1 = A \to B, \quad Y_2 = B, \quad Z = A.$$

It is clear that

$$T(Y_1) = (1 - \alpha_1) \vee \alpha_2 = a,$$

$$T(Y_2) = \alpha_2 = b,$$

$$T(Z) = \alpha_1.$$

In this case, the uncertain entailment model (7.8) becomes

$$\begin{cases} \min |\alpha_1 - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad (1 - \alpha_1) \vee \alpha_2 = a \\ \qquad \alpha_2 = b. \end{cases} \tag{7.18}$$

When $a > b$, there is a unique feasible solution and then the optimal solution is

$$\alpha_1^* = 1 - a, \quad \alpha_2^* = b.$$

Thus $T(A) = \alpha_1^* = 1 - a$. When $a = b$, the feasible set is $[1 - a, 1] \times \{b\}$ and the optimal solution is

$$\alpha_1^* = (1 - a) \vee 0.5, \quad \alpha_2^* = b.$$

Thus $T(A) = \alpha_1^* = (1-a) \vee 0.5$. When $a < b$, there is no feasible solution and the truth values are ill-assigned. In summary, from

$$T(A \to B) = a, \quad T(B) = b \tag{7.19}$$

we entail

$$T(A) = \begin{cases} 1-a, & \text{if } a > b \\ (1-a) \vee 0.5, & \text{if } a = b \\ \text{illness}, & \text{if } a < b. \end{cases} \tag{7.20}$$

This result coincides with the classical modus tollens that if $A \to B$ is true and B is false, then A is false.

7.4 Uncertain Hypothetical Syllogism

Uncertain hypothetical syllogism was presented by Liu [80]. Let A, B, C be independent uncertain propositions. Assume $A \to B$ and $B \to C$ have truth values a and b, respectively. What is the truth value of $A \to C$? Denote the truth values of A, B, C by $\alpha_1, \alpha_2, \alpha_3$, respectively, and write

$$Y_1 = A \to B, \quad Y_2 = B \to C, \quad Z = A \to C.$$

It is clear that

$$T(Y_1) = (1-\alpha_1) \vee \alpha_2 = a,$$

$$T(Y_2) = (1-\alpha_2) \vee \alpha_3 = b,$$

$$T(Z) = (1-\alpha_1) \vee \alpha_3.$$

In this case, the uncertain entailment model (7.8) becomes

$$\begin{cases} \min |(1-\alpha_1) \vee \alpha_3 - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad 0 \le \alpha_3 \le 1 \\ \qquad (1-\alpha_1) \vee \alpha_2 = a \\ \qquad (1-\alpha_2) \vee \alpha_3 = b. \end{cases} \tag{7.21}$$

Write the optimal solution by $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$. When $a \wedge b \ge 0.5$, we have

$$T(A \to C) = (1-\alpha_1^*) \vee \alpha_3^* = a \wedge b.$$

When $a + b \ge 1$ and $a \wedge b < 0.5$, we have

$$T(A \to C) = (1-\alpha_1^*) \vee \alpha_3^* = 0.5.$$

When $a + b < 1$, there is no feasible solution and the truth values are ill-assigned. In summary, from

$$T(A \to B) = a, \quad T(B \to C) = b \tag{7.22}$$

we entail

$$T(A \to C) = \begin{cases} a \wedge b, & \text{if } a \ge 0.5 \text{ and } b \ge 0.5 \\ 0.5, & \text{if } a + b \ge 1 \text{ and } a \wedge b < 0.5 \\ \text{illness}, & \text{if } a + b < 1. \end{cases} \tag{7.23}$$

This result coincides with the classical hypothetical syllogism that if both $A \to B$ and $B \to C$ are true, then $A \to C$ is true.

7.5 Bibliographic Notes

Uncertain entailment was proposed by Liu [80] for determining the truth value of an uncertain proposition via the maximum uncertainty principle when the truth values of other uncertain propositions are given.

From the uncertain entailment model, Liu [80] deduced uncertain modus ponens, uncertain modus tollens, and uncertain hypothetical syllogism. After that, Yang-Gao-Ni [162] investigated the uncertain resolution principle.

Chapter 8

Uncertain Set

Uncertain set was first proposed by Liu [81] in 2010 for modelling unsharp concepts. This chapter will introduce the concepts of uncertain set, membership function, independence, expected value, variance, distance, and entropy. This chapter will also introduce the operational law for uncertain sets via membership functions or inverse membership functions. Finally, conditional uncertain set and conditional membership function are documented.

8.1 Uncertain Set

Roughly speaking, an uncertain set is a set-valued function on an uncertainty space, and attempts to model "unsharp concepts" that are essentially sets but their boundaries are not sharply described (because of the ambiguity of human language). Some typical examples include "young", "tall", "warm", and "most". A formal definition is given as follows.

Definition 8.1 *(Liu [81]) An uncertain set is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to a collection of sets of real numbers such that both $\{B \subset \xi\}$ and $\{\xi \subset B\}$ are events for any Borel set B of real numbers.*

Remark 8.1: Note that the events $\{B \subset \xi\}$ and $\{\xi \subset B\}$ are subsets of the universal set Γ, i.e.,

$$\{B \subset \xi\} = \{\gamma \in \Gamma \mid B \subset \xi(\gamma)\}, \tag{8.1}$$

$$\{\xi \subset B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \subset B\}. \tag{8.2}$$

Remark 8.2: It is clear that uncertain set (Liu [81]) is very different from random set (Robbins [128] and Matheron [112]) and fuzzy set (Zadeh [189]). The essential difference among them is that different measures are used, i.e., random set uses probability measure, fuzzy set uses possibility measure and uncertain set uses uncertain measure.

Remark 8.3: What is the difference between uncertain variable and uncertain set? Both of them belong to the same broad category of uncertain concepts. However, they are differentiated by their mathematical definitions: the former refers to one value, while the latter to a collection of values. Essentially, the difference between uncertain variable and uncertain set focuses on the property of *exclusivity*. If the concept has exclusivity, then it is an uncertain variable. Otherwise, it is an uncertain set. Consider the statement "John is a young man". If we are interested in John's real age, then "young" is an uncertain variable because it is an exclusive concept (John's age cannot be more than one value). For example, if John is 20 years old, then it is impossible that John is 25 years old. In other words, "John is 20 years old" does exclude the possibility that "John is 25 years old". By contrast, if we are interested in what ages can be regarded "young", then "young" is an uncertain set because the concept now has no exclusivity. For example, both 20-year-old and 25-year-old men can be considered "young". In other words, "a 20-year-old man is young" does not exclude the possibility that "a 25-year-old man is young".

Example 8.1: Take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2,\gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\}=0.6$, $\mathcal{M}\{\gamma_2\}=0.3$, $\mathcal{M}\{\gamma_3\}=0.2$. Then

$$\xi(\gamma)=\begin{cases}[1,3], & \text{if } \gamma=\gamma_1\\ [2,4], & \text{if } \gamma=\gamma_2\\ [3,5], & \text{if } \gamma=\gamma_3\end{cases} \tag{8.3}$$

is an uncertain set. See Figure 8.1. Furthermore, we have

$$\mathcal{M}\{2\in\xi\}=\mathcal{M}\{\gamma\,|\,2\in\xi(\gamma)\}=\mathcal{M}\{\gamma_1,\gamma_2\}=0.8, \tag{8.4}$$

$$\mathcal{M}\{[3,4]\subset\xi\}=\mathcal{M}\{\gamma\,|\,[3,4]\subset\xi(\gamma)\}=\mathcal{M}\{\gamma_2,\gamma_3\}=0.4, \tag{8.5}$$

$$\mathcal{M}\{\xi\subset[1,5]\}=\mathcal{M}\{\gamma\,|\,\xi(\gamma)\subset[1,5]\}=\mathcal{M}\{\gamma_1,\gamma_2,\gamma_3\}=1. \tag{8.6}$$

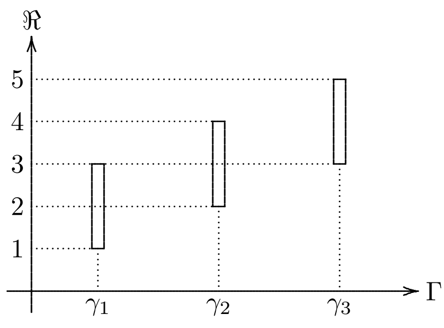


Figure 8.1: An Uncertain Set

Example 8.2: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Then

$$\xi(\gamma) = [0, 3\gamma], \quad \forall \gamma \in \Gamma \tag{8.7}$$

is an uncertain set. Furthermore, we have

$$\mathcal{M}\{2 \in \xi\} = \mathcal{M}\{\gamma \,|\, 2 \in \xi(\gamma)\} = \mathcal{M}\{[2/3, 1]\} = 1/3, \tag{8.8}$$

$$\mathcal{M}\{[0,1] \subset \xi\} = \mathcal{M}\{\gamma \,|\, [0,1] \subset \xi(\gamma)\} = \mathcal{M}\{[1/3, 1]\} = 2/3, \tag{8.9}$$

$$\mathcal{M}\{\xi \subset [0,3)\} = \mathcal{M}\{\gamma \,|\, \xi(\gamma) \subset [0,3)\} = \mathcal{M}\{[0,1)\} = 1. \tag{8.10}$$

Example 8.3: A crisp set A of real numbers is a special uncertain set on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ defined by

$$\xi(\gamma) \equiv A, \quad \forall \gamma \in \Gamma. \tag{8.11}$$

Furthermore, for any Borel set B of real numbers, we have

$$\mathcal{M}\{B \subset \xi\} = \mathcal{M}\{\gamma \,|\, B \subset \xi(\gamma)\} = \mathcal{M}\{\Gamma\} = 1, \quad \text{if } B \subset A, \tag{8.12}$$

$$\mathcal{M}\{B \subset \xi\} = \mathcal{M}\{\gamma \,|\, B \subset \xi(\gamma)\} = \mathcal{M}\{\emptyset\} = 0, \quad \text{if } B \not\subset A, \tag{8.13}$$

$$\mathcal{M}\{\xi \subset B\} = \mathcal{M}\{\gamma \,|\, \xi(\gamma) \subset B\} = \mathcal{M}\{\Gamma\} = 1, \quad \text{if } A \subset B, \tag{8.14}$$

$$\mathcal{M}\{\xi \subset B\} = \mathcal{M}\{\gamma \,|\, \xi(\gamma) \subset B\} = \mathcal{M}\{\emptyset\} = 0, \quad \text{if } A \not\subset B. \tag{8.15}$$

Example 8.4: Let ξ be an uncertain set and let x be a real number. Then

$$\{x \in \xi\}^c = \{\gamma \,|\, x \in \xi(\gamma)\}^c = \{\gamma \,|\, x \notin \xi(\gamma)\} = \{x \notin \xi\}.$$

Thus $\{x \in \xi\}$ and $\{x \notin \xi\}$ are opposite events. Furthermore, by the duality axiom, we obtain

$$\mathcal{M}\{x \in \xi\} + \mathcal{M}\{x \notin \xi\} = 1. \tag{8.16}$$

Exercise 8.1: Let ξ be an uncertain set and let B be a Borel set of real numbers. Show that $\{B \subset \xi\}$ and $\{B \not\subset \xi\}$ are opposite events, and

$$\mathcal{M}\{B \subset \xi\} + \mathcal{M}\{B \not\subset \xi\} = 1. \tag{8.17}$$

Exercise 8.2: Let ξ be an uncertain set and let B be a Borel set of real numbers. Show that $\{\xi \subset B\}$ and $\{\xi \not\subset B\}$ are opposite events, and

$$\mathcal{M}\{\xi \subset B\} + \mathcal{M}\{\xi \not\subset B\} = 1. \tag{8.18}$$

Exercise 8.3: Let ξ and η be two uncertain sets. Show that $\{\xi \subset \eta\}$ and $\{\xi \not\subset \eta\}$ are opposite events, and

$$\mathcal{M}\{\xi \subset \eta\} + \mathcal{M}\{\xi \not\subset \eta\} = 1. \tag{8.19}$$

Exercise 8.4: Let $\emptyset$ be the empty set, and let ξ be an uncertain set. Show that

$$\mathcal{M}\{\emptyset \subset \xi\} = 1. \tag{8.20}$$

Exercise 8.5: Let ξ be an uncertain set, and let $\Re$ be the set of real numbers. Show that

$$\mathcal{M}\{\xi \subset \Re\} = 1. \tag{8.21}$$

Exercise 8.6: Let ξ be an uncertain set. Show that ξ is always included in itself, i.e.,

$$\mathcal{M}\{\xi \subset \xi\} = 1. \tag{8.22}$$

Theorem 8.1 *(Liu [98], Fundamental Relationship) Let ξ be an uncertain set, and let B be a crisp set of real numbers. Then*

$$\{B \subset \xi\} = \bigcap_{x \in B} \{x \in \xi\}, \tag{8.23}$$

$$\{\xi \subset B\} = \bigcap_{x \in B^c} \{x \notin \xi\}. \tag{8.24}$$

Proof: For any $\gamma \in \{B \subset \xi\}$, we have $B \subset \xi(\gamma)$. Thus $x \in \xi(\gamma)$ whenever $x \in B$. This means $\gamma \in \{x \in \xi\}$ and then $\{B \subset \xi\} \subset \{x \in \xi\}$ for any $x \in B$. Hence

$$\{B \subset \xi\} \subset \bigcap_{x \in B} \{x \in \xi\}. \tag{8.25}$$

On the other hand, for any

$$\gamma \in \bigcap_{x \in B} \{x \in \xi\},$$

we have $x \in \xi(\gamma)$ whenever $x \in B$. Thus $B \subset \xi(\gamma)$, i.e., $\gamma \in \{B \subset \xi\}$. This means

$$\{B \subset \xi\} \supset \bigcap_{x \in B} \{x \in \xi\}. \tag{8.26}$$

It follows from (8.25) and (8.26) that (8.23) holds. The first equation is proved. Next we verify the second equation. For any $\gamma \in \{\xi \subset B\}$, we have $\xi(\gamma) \subset B$. Thus $x \notin \xi(\gamma)$ whenever $x \in B^c$. This means $\gamma \in \{x \notin \xi\}$ and then $\{\xi \subset B\} \subset \{x \notin \xi\}$ for any $x \in B^c$. Hence

$$\{\xi \subset B\} \subset \bigcap_{x \in B^c} \{x \notin \xi\}. \tag{8.27}$$

On the other hand, for any

$$\gamma \in \bigcap_{x \in B^c} \{x \notin \xi\},$$

we have $x \notin \xi(\gamma)$ whenever $x \in B^c$. Thus $\xi(\gamma) \subset B$, i.e., $\gamma \in \{\xi \subset B\}$. This means

$$\{\xi \subset B\} \supset \bigcap_{x \in B^c} \{x \notin \xi\}. \tag{8.28}$$

It follows from (8.27) and (8.28) that (8.24) holds. The theorem is proved.

Definition 8.2 *An uncertain set ξ on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be (i) nonempty if*

$$\xi(\gamma) \neq \emptyset \tag{8.29}$$

for almost all $\gamma \in \Gamma$, (ii) empty if

$$\xi(\gamma) = \emptyset \tag{8.30}$$

for almost all $\gamma \in \Gamma$, and (iii) half-empty if otherwise.

Example 8.5: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Then

$$\xi(\gamma) = [0, \gamma], \quad \forall \gamma \in \Gamma \tag{8.31}$$

is a nonempty uncertain set,

$$\xi(\gamma) = \emptyset, \quad \forall \gamma \in \Gamma \tag{8.32}$$

is an empty uncertain set, and

$$\xi(\gamma) = \begin{cases} \emptyset, & \text{if } \gamma > 0.8 \\ [0, \gamma], & \text{if } \gamma \le 0.8 \end{cases} \tag{8.33}$$

is a half-empty uncertain set.

Union, Intersection and Complement

Definition 8.3 *Let ξ and η be two uncertain sets on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then (i) the union $\xi \cup \eta$ of the uncertain sets ξ and η is*

$$(\xi \cup \eta)(\gamma) = \xi(\gamma) \cup \eta(\gamma), \quad \forall \gamma \in \Gamma; \tag{8.34}$$

(ii) the intersection $\xi \cap \eta$ of the uncertain sets ξ and η is

$$(\xi \cap \eta)(\gamma) = \xi(\gamma) \cap \eta(\gamma), \quad \forall \gamma \in \Gamma; \tag{8.35}$$

(iii) the complement ξ^c of the uncertain set ξ is

$$\xi^c(\gamma) = \xi(\gamma)^c, \quad \forall \gamma \in \Gamma. \tag{8.36}$$

Example 8.6: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$. Let ξ and η be two uncertain sets,

$$\xi(\gamma) = \begin{cases} [1,2], & \text{if } \gamma = \gamma_1 \\ [1,3], & \text{if } \gamma = \gamma_2 \\ [1,4], & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} (2,3), & \text{if } \gamma = \gamma_1 \\ (2,4), & \text{if } \gamma = \gamma_2 \\ (2,5), & \text{if } \gamma = \gamma_3. \end{cases}$$

Then their union is

$$(\xi \cup \eta)(\gamma) = \begin{cases} [1,3), & \text{if } \gamma = \gamma_1 \\ [1,4), & \text{if } \gamma = \gamma_2 \\ [1,5), & \text{if } \gamma = \gamma_3, \end{cases}$$

their intersection is

$$(\xi \cap \eta)(\gamma) = \begin{cases} \emptyset, & \text{if } \gamma = \gamma_1 \\ (2,3], & \text{if } \gamma = \gamma_2 \\ (2,4], & \text{if } \gamma = \gamma_3, \end{cases}$$

and their complement sets are

$$\xi^c(\gamma) = \begin{cases} (-\infty,1) \cup (2,+\infty), & \text{if } \gamma = \gamma_1 \\ (-\infty,1) \cup (3,+\infty), & \text{if } \gamma = \gamma_2 \\ (-\infty,1) \cup (4,+\infty), & \text{if } \gamma = \gamma_3, \end{cases}$$

$$\eta^c(\gamma) = \begin{cases} (-\infty,2] \cup [3,+\infty), & \text{if } \gamma = \gamma_1 \\ (-\infty,2] \cup [4,+\infty), & \text{if } \gamma = \gamma_2 \\ (-\infty,2] \cup [5,+\infty), & \text{if } \gamma = \gamma_3. \end{cases}$$

Theorem 8.2 *(Idempotent Law) Let ξ be an uncertain set. Then we have*

$$\xi \cup \xi = \xi, \quad \xi \cap \xi = \xi. \tag{8.37}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that the union is

$$(\xi \cup \xi)(\gamma) = \xi(\gamma) \cup \xi(\gamma) = \xi(\gamma).$$

Thus we have $\xi \cup \xi = \xi$. In addition, the intersection is

$$(\xi \cap \xi)(\gamma) = \xi(\gamma) \cap \xi(\gamma) = \xi(\gamma).$$

Thus we have $\xi \cap \xi = \xi$.

Theorem 8.3 *(Double-Negation Law) Let ξ be an uncertain set. Then we have*

$$(\xi^c)^c = \xi. \tag{8.38}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of complement that

$$(\xi^c)^c(\gamma) = (\xi^c(\gamma))^c = (\xi(\gamma)^c)^c = \xi(\gamma).$$

Thus we have $(\xi^c)^c = \xi$.

Theorem 8.4 *(Law of Excluded Middle and Law of Contradiction) Let ξ be an uncertain set and let ξ^c be its complement. Then*

$$\xi \cup \xi^c \equiv \Re, \quad \xi \cap \xi^c \equiv \emptyset. \tag{8.39}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that the union is

$$(\xi \cup \xi^c)(\gamma) = \xi(\gamma) \cup \xi^c(\gamma) = \xi(\gamma) \cup \xi(\gamma)^c \equiv \Re.$$

Thus we have $\xi \cup \xi^c \equiv \Re$. In addition, the intersection is

$$(\xi \cap \xi^c)(\gamma) = \xi(\gamma) \cap \xi^c(\gamma) = \xi(\gamma) \cap \xi(\gamma)^c \equiv \emptyset.$$

Thus we have $\xi \cap \xi^c \equiv \emptyset$.

Theorem 8.5 *(Commutative Law) Let ξ and η be uncertain sets. Then we have*

$$\xi \cup \eta = \eta \cup \xi, \quad \xi \cap \eta = \eta \cap \xi. \tag{8.40}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that

$$(\xi \cup \eta)(\gamma) = \xi(\gamma) \cup \eta(\gamma) = \eta(\gamma) \cup \xi(\gamma) = (\eta \cup \xi)(\gamma).$$

Thus we have $\xi \cup \eta = \eta \cup \xi$. In addition, it follows that

$$(\xi \cap \eta)(\gamma) = \xi(\gamma) \cap \eta(\gamma) = \eta(\gamma) \cap \xi(\gamma) = (\eta \cap \xi)(\gamma).$$

Thus we have $\xi \cap \eta = \eta \cap \xi$.

Theorem 8.6 *(Associative Law) Let ξ, η, τ be uncertain sets. Then we have*

$$(\xi \cup \eta) \cup \tau = \xi \cup (\eta \cup \tau), \quad (\xi \cap \eta) \cap \tau = \xi \cap (\eta \cap \tau). \tag{8.41}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that

$$\begin{aligned} ((\xi \cup \eta) \cup \tau)(\gamma) &= (\xi(\gamma) \cup \eta(\gamma)) \cup \tau(\gamma) \\ &= \xi(\gamma) \cup (\eta(\gamma) \cup \tau(\gamma)) = (\xi \cup (\eta \cup \tau))(\gamma). \end{aligned}$$

Thus we have $(\xi \cup \eta) \cup \tau = \xi \cup (\eta \cup \tau)$. In addition, it follows that

$$\begin{aligned} ((\xi \cap \eta) \cap \tau)(\gamma) &= (\xi(\gamma) \cap \eta(\gamma)) \cap \tau(\gamma) \\ &= \xi(\gamma) \cap (\eta(\gamma) \cap \tau(\gamma)) = (\xi \cap (\eta \cap \tau))(\gamma). \end{aligned}$$

Thus we have $(\xi \cap \eta) \cap \tau = \xi \cap (\eta \cap \tau)$.

Theorem 8.7 *(Distributive Law) Let ξ,η,τ be uncertain sets. Then we have*

$$\xi\cup(\eta\cap\tau)=(\xi\cup\eta)\cap(\xi\cup\tau),\quad \xi\cap(\eta\cup\tau)=(\xi\cap\eta)\cup(\xi\cap\tau). \tag{8.42}$$

Proof: For each $\gamma\in\Gamma$, it follows from the definition of uncertain set that

$$\begin{aligned}(\xi\cup(\eta\cap\tau))(\gamma)&=\xi(\gamma)\cup(\eta(\gamma)\cap\tau(\gamma))\\&=(\xi(\gamma)\cup\eta(\gamma))\cap(\xi(\gamma)\cup\tau(\gamma))\\&=((\xi\cup\eta)\cap(\xi\cup\tau))(\gamma).\end{aligned}$$

Thus we have $\xi\cup(\eta\cap\tau)=(\xi\cup\eta)\cap(\xi\cup\tau)$. In addition, it follows that

$$\begin{aligned}(\xi\cap(\eta\cup\tau))(\gamma)&=\xi(\gamma)\cap(\eta(\gamma)\cup\tau(\gamma))\\&=(\xi(\gamma)\cap\eta(\gamma))\cup(\xi(\gamma)\cap\tau(\gamma))\\&=((\xi\cap\eta)\cup(\xi\cap\tau))(\gamma).\end{aligned}$$

Thus we have $\xi\cap(\eta\cup\tau)=(\xi\cap\eta)\cup(\xi\cap\tau)$.

Theorem 8.8 *(Absorbtion Law) Let ξ and η be uncertain sets. Then we have*

$$\xi\cup(\xi\cap\eta)=\xi,\quad \xi\cap(\xi\cup\eta)=\xi. \tag{8.43}$$

Proof: For each $\gamma\in\Gamma$, it follows from the definition of uncertain set that

$$(\xi\cup(\xi\cap\eta))(\gamma)=\xi(\gamma)\cup(\xi(\gamma)\cap\eta(\gamma))=\xi(\gamma).$$

Thus we have $\xi\cup(\xi\cap\eta)=\xi$. In addition, since

$$(\xi\cap(\xi\cup\eta))(\gamma)=\xi(\gamma)\cap(\xi(\gamma)\cup\eta(\gamma))=\xi(\gamma),$$

we get $\xi\cap(\xi\cup\eta)=\xi$.

Theorem 8.9 *(De Morgan's Law) Let ξ and η be uncertain sets. Then we have*

$$(\xi\cup\eta)^c=\xi^c\cap\eta^c,\quad (\xi\cap\eta)^c=\xi^c\cup\eta^c. \tag{8.44}$$

Proof: For each $\gamma\in\Gamma$, it follows from the definition of complement that

$$(\xi\cup\eta)^c(\gamma)=((\xi(\gamma)\cup\eta(\gamma))^c=\xi(\gamma)^c\cap\eta(\gamma)^c=(\xi^c\cap\eta^c)(\gamma).$$

Thus we have $(\xi\cup\eta)^c=\xi^c\cap\eta^c$. In addition, since

$$(\xi\cap\eta)^c(\gamma)=((\xi(\gamma)\cap\eta(\gamma))^c=\xi(\gamma)^c\cup\eta(\gamma)^c=(\xi^c\cup\eta^c)(\gamma),$$

we get $(\xi\cap\eta)^c=\xi^c\cup\eta^c$.

Exercise 8.7: Let ξ be an uncertain set and let x be a real number. Show that

$$\{x \in \xi^c\} = \{x \notin \xi\} \tag{8.45}$$

and

$$\mathcal{M}\{x \in \xi^c\} = \mathcal{M}\{x \notin \xi\}. \tag{8.46}$$

Exercise 8.8: Let ξ be an uncertain set and let x be a real number. Show that $\{x \in \xi\}$ and $\{x \in \xi^c\}$ are opposite events, and

$$\mathcal{M}\{x \in \xi\} + \mathcal{M}\{x \in \xi^c\} = 1. \tag{8.47}$$

Exercise 8.9: Let ξ be an uncertain set and let B be a Borel set of real numbers. Show that $\{B \subset \xi\}$ and $\{B \subset \xi^c\}$ are not necessarily opposite events.

Exercise 8.10: Let ξ and η be two uncertain sets. Show that $\{\xi \subset \eta\}$ and $\{\eta^c \subset \xi^c\}$ are identical events, i.e.,

$$\{\xi \subset \eta\} = \{\eta^c \subset \xi^c\}. \tag{8.48}$$

Exercise 8.11: Let ξ and η be two uncertain sets. Show that $\{\xi \subset \eta\}$ and $\{\xi \subset \eta^c\}$ are not necessarily opposite events.

Function of Uncertain Sets

Definition 8.4 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain sets on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, and let f be a measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain set defined by*

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \cdots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma. \tag{8.49}$$

Example 8.7: Let ξ be an uncertain set on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and let A be a crisp set of real numbers. Then $\xi + A$ is also an uncertain set determined by

$$(\xi + A)(\gamma) = \xi(\gamma) + A, \quad \forall \gamma \in \Gamma. \tag{8.50}$$

Example 8.8: Note that the empty set $\emptyset$ annihilates every other set. For example, $A + \emptyset = \emptyset$ and $A \times \emptyset = \emptyset$. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$. Define two uncertain sets,

$$\xi(\gamma) = \begin{cases} \emptyset, & \text{if } \gamma = \gamma_1 \\ [1, 3], & \text{if } \gamma = \gamma_2 \\ [1, 4], & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} (2, 3), & \text{if } \gamma = \gamma_1 \\ (2, 4), & \text{if } \gamma = \gamma_2 \\ (2, 5), & \text{if } \gamma = \gamma_3. \end{cases}$$

Then their sum is

$$(\xi+\eta)(\gamma)=\begin{cases}\emptyset, & \text{if } \gamma=\gamma_1\\ (3,7), & \text{if } \gamma=\gamma_2\\ (3,9), & \text{if } \gamma=\gamma_3,\end{cases}$$

and their product is

$$(\xi\times\eta)(\gamma)=\begin{cases}\emptyset, & \text{if } \gamma=\gamma_1\\ (2,12), & \text{if } \gamma=\gamma_2\\ (2,20), & \text{if } \gamma=\gamma_3.\end{cases}$$

Exercise 8.12: Let ξ be an uncertain set. (i) Show that $\xi+\xi\not\equiv 2\xi$. (ii) Do you think the same of crisp set?

Exercise 8.13: Let ξ be an uncertain set. What are the potential values of the difference $\xi-\xi$?

8.2 Membership Function

It is well-known that a crisp set can be described by its indicator function. As a generalization of indicator function, membership function will be used to describe an uncertain set.

Definition 8.5 *(Liu [87]) An uncertain set ξ is said to have a membership function μ if for any Borel set B of real numbers, we have*

$$\mathcal{M}\{B\subset\xi\}=\inf_{x\in B}\mu(x), \tag{8.51}$$

$$\mathcal{M}\{\xi\subset B\}=1-\sup_{x\in B^c}\mu(x). \tag{8.52}$$

The above equations will be called measure inversion formulas.

Theorem 8.10 *Let ξ be an uncertain set whose membership function μ exists. Then*

$$\mu(x)=\mathcal{M}\{x\in\xi\} \tag{8.53}$$

for any number x.

Proof: For any number x, it follows from the first measure inversion formula that

$$\mathcal{M}\{x\in\xi\}=\mathcal{M}\{\{x\}\subset\xi\}=\inf_{y\in\{x\}}\mu(y)=\mu(x).$$

The theorem is proved.

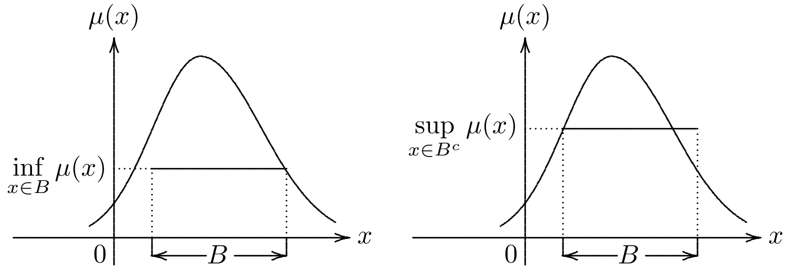


Figure 8.2: $\mathcal{M}\{B \subset \xi\} = \inf_{x\in B} \mu(x)$ and $\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x\in B^c} \mu(x)$

Remark 8.4: The value of $\mu(x)$ is just the membership degree that x belongs to the uncertain set ξ. If $\mu(x) = 1$, then x completely belongs to ξ; if $\mu(x) = 0$, then x does not belong to ξ at all. Thus the larger the value of $\mu(x)$ is, the more true x belongs to ξ.

Theorem 8.11 *Let ξ be an uncertain set with membership function μ. Then*

$$\mathcal{M}\{x \not\in \xi\} = 1 - \mu(x) \tag{8.54}$$

for any number x.

Proof: Since $\{x \not\in \xi\}$ and $\{x \in \xi\}$ are opposite events, it follows from the duality axiom of uncertain measure that

$$\mathcal{M}\{x \not\in \xi\} = 1 - \mathcal{M}\{x \in \xi\} = 1 - \mu(x).$$

The theorem is proved.

Remark 8.5: Theorem 8.11 states that if an element x belongs to an uncertain set with membership degree α, then x does not belong to the uncertain set with membership degree $1 - \alpha$.

Theorem 8.12 *Let ξ be an uncertain set with membership function μ. Then*

$$\mathcal{M}\{x \in \xi^c\} = 1 - \mu(x) \tag{8.55}$$

for any number x.

Proof: Since $\{x \in \xi^c\}$ and $\{x \in \xi\}$ are opposite events, it follows from the duality axiom of uncertain measure that

$$\mathcal{M}\{x \in \xi^c\} = 1 - \mathcal{M}\{x \in \xi\} = 1 - \mu(x).$$

The theorem is proved.

Remark 8.6: Theorem 8.12 states that if an element x belongs to an uncertain set with membership degree α, then x belongs to its complement set with membership degree $1-\alpha$.

Remark 8.7: For any membership function μ, it is clear that $0 \le \mu(x) \le 1$. We will always take

$$\inf_{x\in\emptyset} \mu(x) = 1, \quad \sup_{x\in\emptyset} \mu(x) = 0. \tag{8.56}$$

Thus we have

$$\mathcal{M}\{\emptyset \subset \xi\} = 1 = \inf_{x\in\emptyset} \mu(x).$$

That is, the first measure inversion formula always holds for $B=\emptyset$. Furthermore, we have

$$\mathcal{M}\{\xi \subset \Re\} = 1 = 1 - \sup_{x\in\emptyset} \mu(x).$$

That is, the second measure inversion formula always holds for $B=\Re$.

Example 8.9: The set $\Re$ of real numbers is a special uncertain set $\xi(\gamma)\equiv\Re$. Such an uncertain set has a membership function

$$\mu(x) \equiv 1 \tag{8.57}$$

that is just the indicator function of $\Re$. In order to prove it, we must verify that $\Re$ and μ simultaneously satisfy the two measure inversion formulas (8.51) and (8.52). Let B be a Borel set of real numbers. If $B=\emptyset$, then the first measure inversion formula always holds. If $B\neq\emptyset$, then

$$\mathcal{M}\{B \subset \xi\} = \mathcal{M}\{\Gamma\} = 1 = \inf_{x\in B} \mu(x).$$

The first measure inversion formula is verified. Next we prove the second measure inversion formula. If $B=\Re$, then the second measure inversion formula always holds. If $B\neq\Re$, then

$$\mathcal{M}\{\xi \subset B\} = \mathcal{M}\{\emptyset\} = 0 = 1 - \sup_{x\in B^c} \mu(x).$$

The second measure inversion formula is verified. Therefore, the uncertain set $\xi(\gamma)\equiv\Re$ has a membership function $\mu(x)\equiv 1$.

Exercise 8.14: The empty set $\emptyset$ is a special uncertain set $\xi(\gamma)\equiv\emptyset$. Show that such an uncertain set has a membership function

$$\mu(x) \equiv 0 \tag{8.58}$$

that is just the indicator function of $\emptyset$.

Exercise 8.15: A crisp set A of real numbers is a special uncertain set $\xi(\gamma) \equiv A$. Show that such an uncertain set has a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \tag{8.59}$$

that is just the indicator function of A.

Exercise 8.16: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.4$, $\mathcal{M}\{\gamma_2\} = 0.6$. Show that the uncertain set

$$\xi(\gamma) = \begin{cases} \emptyset, & \text{if } \gamma = \gamma_1 \\ A, & \text{if } \gamma = \gamma_2 \end{cases}$$

has a membership function

$$\mu(x) = \begin{cases} 0.6, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \tag{8.60}$$

where A is a crisp set of real numbers.

Exercise 8.17: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. (i) Show that the uncertain set

$$\xi(\gamma) = [-\gamma, \gamma], \quad \forall \gamma \in [0, 1] \tag{8.61}$$

has a membership function

$$\mu(x) = \begin{cases} 1 - |x|, & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{8.62}$$

(ii) What is the membership function of $\xi(\gamma) = [\gamma - 1, 1 - \gamma]$? (iii) What do those two uncertain sets make you think about? (iv) Design a third uncertain set whose membership function is also (8.62).

Exercise 8.18: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$. Define an uncertain set

$$\xi(\gamma) = \begin{cases} [2, 3], & \text{if } \gamma = \gamma_1 \\ [0, 5], & \text{if } \gamma = \gamma_2 \\ [1, 4], & \text{if } \gamma = \gamma_3. \end{cases}$$

(i) What is the membership function of ξ? (ii) Please justify your answer. (Hint: If ξ does have a membership function, then $\mu(x) = \mathcal{M}\{x \in \xi\}$.)

Exercise 8.19: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Define an uncertain set

$$\xi(\gamma) = \left(\gamma^2, +\infty\right). \tag{8.63}$$

(i) What is the membership function of ξ? (ii) What is the membership function of the complement set ξ^c? (iii) What do those two uncertain sets make you think about?

Exercise 8.20: It is not true that every uncertain set has a membership function. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.4$, $\mathcal{M}\{\gamma_2\} = 0.6$. Show that the uncertain set

$$\xi(\gamma) = \begin{cases} [1,3], & \text{if } \gamma = \gamma_1 \\ [2,4], & \text{if } \gamma = \gamma_2 \end{cases} \tag{8.64}$$

has no membership function. (Hint: If ξ does have a membership function, then by using $\mu(x) = \mathcal{M}\{x \in \xi\}$, we get

$$\mu(x) = \begin{cases} 0.4, & \text{if } 1 \le x < 2 \\ 1, & \text{if } 2 \le x \le 3 \\ 0.6, & \text{if } 3 < x \le 4 \\ 0, & \text{otherwise.} \end{cases} \tag{8.65}$$

Verify that ξ and μ cannot simultaneously satisfy the two measure inversion formulas (8.51) and (8.52).)

Exercise 8.21: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Show that the uncertain set

$$\xi(\gamma) = [\gamma, \gamma+1], \quad \forall \gamma \in \Gamma \tag{8.66}$$

has no membership function.

Definition 8.6 *An uncertain set ξ is called triangular if it has a membership function*

$$\mu(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ \dfrac{x-c}{b-c}, & \text{if } b \le x \le c \end{cases} \tag{8.67}$$

denoted by (a, b, c) where a, b, c are real numbers with $a < b < c$.

Definition 8.7 *An uncertain set ξ is called trapezoidal if it has a membership function*

$$\mu(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } b \le x \le c \\ \dfrac{x-d}{c-d}, & \text{if } c \le x \le d \end{cases} \tag{8.68}$$

denoted by (a, b, c, d) where a, b, c, d are real numbers with $a < b < c < d$.

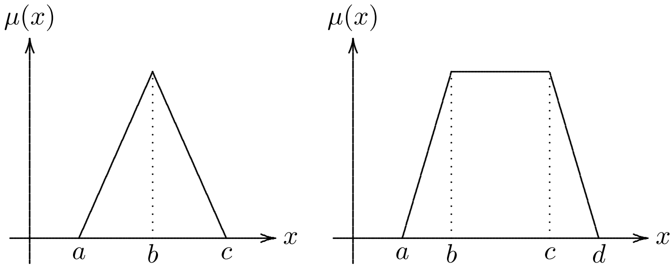


Figure 8.3: Triangular and Trapezoidal Membership Functions

What is "young"?

Sometimes we say "those students are young". What ages can be considered "young"? In this case, "young" may be regarded as an uncertain set whose membership function is

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 35 \\ (45-x)/10, & \text{if } 35 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{8.69}$$

Note that we do not say "young" if the age is below 15.

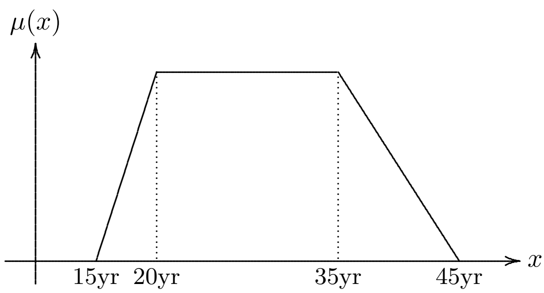


Figure 8.4: Membership Function of "young"

What is "tall"?

Sometimes we say "those sportsmen are tall". What heights (centimeters) can be considered "tall"? In this case, "tall" may be regarded as an uncertain set whose membership function is

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x-180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } 185 \le x \le 195 \\ (200-x)/5, & \text{if } 195 \le x \le 200 \\ 0, & \text{if } x \ge 200. \end{cases} \tag{8.70}$$

Note that we do not say "tall" if the height is over 200cm.

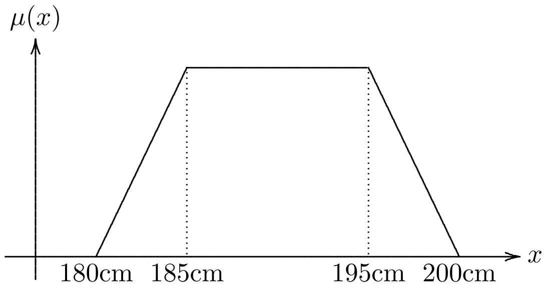


Figure 8.5: Membership Function of "tall"

What is "warm"?

Sometimes we say "those days are warm". What temperatures can be considered "warm"? In this case, "warm" may be regarded as an uncertain set whose membership function is

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/3, & \text{if } 15 \le x \le 18 \\ 1, & \text{if } 18 \le x \le 24 \\ (28-x)/4, & \text{if } 24 \le x \le 28 \\ 0, & \text{if } 28 \le x. \end{cases} \tag{8.71}$$

What is "most"?

Sometimes we say "most students are boys". What percentages can be considered "most"? In this case, "most" may be regarded as an uncertain set

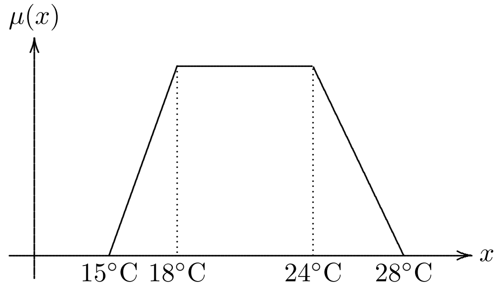


Figure 8.6: Membership Function of "warm"

whose membership function is

$$\mu(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.7 \\ 20(x-0.7), & \text{if } 0.7 \le x \le 0.75 \\ 1, & \text{if } 0.75 \le x \le 0.85 \\ 20(0.9-x), & \text{if } 0.85 \le x \le 0.9 \\ 0, & \text{if } 0.9 \le x \le 1. \end{cases} \tag{8.72}$$

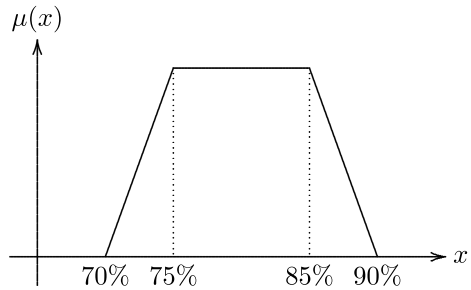


Figure 8.7: Membership Function of "most"

What uncertain sets have membership functions?

It is known that some uncertain sets do not have membership functions. This subsection shows that totally ordered uncertain sets defined on a continuous uncertainty space always have membership functions.

Definition 8.8 *(Liu [98]) An uncertain set* ξ *defined on the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called totally ordered if* $\{\xi(\gamma) \,|\, \gamma \in \Gamma\}$ *is a totally ordered set, i.e., for any given* γ_1 *and* $\gamma_2 \in \Gamma$*, either* $\xi(\gamma_1) \subset \xi(\gamma_2)$ *or* $\xi(\gamma_2) \subset \xi(\gamma_1)$ *holds.*

Example 8.10: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and let A be a crisp set of real numbers. The uncertain set $\xi(\gamma) \equiv A$ is of total order.

Example 8.11: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$. The uncertain set

$$\xi(\gamma) = \begin{cases} [2,3], & \text{if } \gamma = \gamma_1 \\ [0,5], & \text{if } \gamma = \gamma_2 \\ [1,4], & \text{if } \gamma = \gamma_3 \end{cases} \tag{8.73}$$

is of total order.

Example 8.12: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. The uncertain set

$$\xi(\gamma) = [-\gamma, \gamma], \quad \forall \gamma \in \Gamma \tag{8.74}$$

is of total order.

Example 8.13: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. The uncertain set

$$\xi(\gamma) = [\gamma, \gamma + 1], \quad \forall \gamma \in \Gamma \tag{8.75}$$

is not of total order.

Exercise 8.22: Let ξ be a totally ordered uncertain set. Show that its complement ξ^c is also of total order.

Exercise 8.23: Let ξ be a totally ordered uncertain set, and let f be a real-valued function. Show that $f(\xi)$ is also of total order.

Exercise 8.24: Let ξ and η be totally ordered uncertain sets. Show that their union $\xi \cup \eta$ is not necessarily of total order.

Exercise 8.25: Let ξ and η be totally ordered uncertain sets. Show that their intersection $\xi \cap \eta$ is not necessarily of total order.

Theorem 8.13 *(Liu [98]) Let ξ be a totally ordered uncertain set, and let B be a crisp set of real numbers. Then (i) the collection $\{x \in \xi\}$ indexed by $x \in B$ is of total order, and (ii) the collection $\{x \notin \xi\}$ indexed by $x \in B$ is also of total order.*

Proof: If $\{x \in \xi\}$ indexed by $x \in B$ is not of total order, then there exist two numbers x_1 and x_2 in B such that neither $\{x_1 \in \xi\} \subset \{x_2 \in \xi\}$ nor $\{x_2 \in \xi\} \subset \{x_1 \in \xi\}$ holds. This means there exist γ_1 and γ_2 in Γ such that

$$\gamma_1 \in \{x_1 \in \xi\}, \quad \gamma_1 \notin \{x_2 \in \xi\},$$

$$\gamma_2 \in \{x_2 \in \xi\}, \quad \gamma_2 \notin \{x_1 \in \xi\}.$$

That is,

$$x_1 \in \xi(\gamma_1), \quad x_1 \notin \xi(\gamma_2),$$

$$x_2 \in \xi(\gamma_2), \quad x_2 \notin \xi(\gamma_1).$$

Thus neither $\xi(\gamma_1) \subset \xi(\gamma_2)$ nor $\xi(\gamma_2) \subset \xi(\gamma_1)$ holds. This result is in contradiction with that ξ is a totally ordered uncertain set. Therefore, $\{x \in \xi\}$ indexed by $x \in B$ is of total order. The first part is proved. It follows from

$$\{x \notin \xi\} = \{x \in \xi\}^c$$

that $\{x \notin \xi\}$ indexed by $x \in B$ is also of total order. The second part is verified.

Theorem 8.14 *(Liu [98], Existence Theorem) Let ξ be a totally ordered uncertain set on a continuous uncertainty space. Then its membership function always exists, and*

$$\mu(x) = \mathcal{M}\{x \in \xi\}. \tag{8.76}$$

Proof: In order to prove that μ is the membership function of ξ, we must verify the two measure inversion formulas. Let B be any Borel set of real numbers. Theorem 8.1 states that

$$\{B \subset \xi\} = \bigcap_{x \in B} \{x \in \xi\}.$$

Since the uncertain measure is assumed to be continuous, and $\{x \in \xi\}$ indexed by $x \in B$ is of total order, we obtain

$$\mathcal{M}\{B \subset \xi\} = \mathcal{M}\left\{\bigcap_{x \in B} (x \in \xi)\right\} = \inf_{x \in B} \mathcal{M}\{x \in \xi\} = \inf_{x \in B} \mu(x).$$

The first measure inversion formula is verified. Next, Theorem 8.1 states that

$$\{\xi \subset B\} = \bigcap_{x \in B^c} \{x \notin \xi\}.$$

Since the uncertain measure is assumed to be continuous, and $\{x \notin \xi\}$ indexed by $x \in B^c$ is of total order, we obtain

$$\mathcal{M}\{\xi \subset B\} = \mathcal{M}\left\{\bigcap_{x \in B^c} (x \notin \xi)\right\} = \inf_{x \in B^c} \mathcal{M}\{x \notin \xi\} = 1 - \sup_{x \in B^c} \mu(x).$$

The second measure inversion formula is verified. Therefore, μ is the membership function of ξ.

Remark 8.8: Theorem 8.14 tells us that the membership function of a totally ordered uncertain set on a continuous uncertainty space exists and is determined by $\mu(x) = \mathcal{M}\{x \in \xi\}$. In other words, the two measure inversion formulas are no longer required to be verified whenever the uncertain set is of total order and defined on a continuous uncertainty space.

Example 8.14: The continuity condition in Theorem 8.14 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $(0, 1)$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise.} \end{cases} \tag{8.77}$$

Then

$$\xi(\gamma) = (-\gamma, \gamma), \quad \forall \gamma \in (0, 1) \tag{8.78}$$

is a totally ordered uncertain set on a discontinuous uncertainty space. If it indeed has a membership function, then

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } -1 < x < 0 \text{ or } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{8.79}$$

However,

$$\mathcal{M}\{(-1, 1) \subset \xi\} = \mathcal{M}\{\emptyset\} = 0 \neq 0.5 = \inf_{x \in (-1,1)} \mu(x). \tag{8.80}$$

That is, the first measure inversion formula is not valid and then ξ has no membership function. Therefore, the continuity condition cannot be removed.

Example 8.15: Some non-totally ordered uncertain sets may have membership functions. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise.} \end{cases} \tag{8.81}$$

Then

$$\xi(\gamma) = \begin{cases} \{1\}, & \text{if } \gamma = \gamma_1 \\ \{1, 2\}, & \text{if } \gamma = \gamma_2 \\ \{1, 3\}, & \text{if } \gamma = \gamma_3 \\ \{1, 2, 3\}, & \text{if } \gamma = \gamma_4 \end{cases} \tag{8.82}$$

is a non-totally ordered uncertain set. However, it has a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0.5, & \text{if } x = 2 \text{ or } 3 \\ 0, & \text{otherwise} \end{cases} \tag{8.83}$$

because ξ and μ can simultaneously satisfy the two measure inversion formulas (8.51) and (8.52).

Remark 8.9: In practice, the unsharp concepts like "young", "tall", "warm", and "most" can be regarded as totally ordered uncertain sets on a continuous uncertainty space.

Sufficient and Necessary Condition

Theorem 8.15 *(Liu [84]) A real-valued function μ is a membership function of uncertain set if and only if*

$$0 \le \mu(x) \le 1. \tag{8.84}$$

Proof: If μ is a membership function of some uncertain set ξ, then $\mu(x) = \mathcal{M}\{x \in \xi\}$ and $0 \le \mu(x) \le 1$. Conversely, suppose μ is a function such that $0 \le \mu(x) \le 1$. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Then

$$\xi(\gamma) = \{x \in \Re \mid \mu(x) \ge \gamma\} \tag{8.85}$$

is a totally ordered uncertain set defined on the continuous uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. See Figure 8.8. By using Theorem 8.14, it is easy to verify that ξ has the membership function μ.

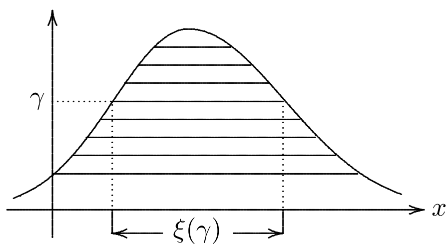


Figure 8.8: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Then $\xi(\gamma) = \{x \in \Re \mid \mu(x) \ge \gamma\}$ has the membership function μ. Keep in mind that ξ is not the unique uncertain set whose membership function is μ.

Example 8.16: Let c be a number between 0 and 1. It follows from the sufficient and necessary condition that

$$\mu(x) \equiv c \tag{8.86}$$

is a membership function. Take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Define

$$\xi(\gamma)=\begin{cases}\Re, & \text{if } 0\le\gamma\le c\\ \emptyset, & \text{if } c<\gamma\le 1.\end{cases} \tag{8.87}$$

It is easy to verify that ξ is a totally ordered uncertain set on a continuous uncertainty space, and has the membership function μ.

Example 8.17: Let us design an uncertain set whose membership function is

$$\mu(x)=\exp(-x^2) \tag{8.88}$$

for any real number x. Take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Define

$$\xi(\gamma)=(-\sqrt{-\ln\gamma},\sqrt{-\ln\gamma}),\quad \forall\gamma\in[0,1]. \tag{8.89}$$

It is easy to verify that ξ is a totally ordered uncertain set on a continuous uncertainty space, and has the membership function μ.

Exercise 8.26: Design an uncertain set whose membership function is just

$$\mu(x)=\frac{1}{2}\exp(-x^2) \tag{8.90}$$

for any real number x.

Exercise 8.27: Design an uncertain set whose membership function is just

$$\mu(x)=\frac{1}{2}\exp(-x^2)+\frac{1}{2} \tag{8.91}$$

for any real number x.

Theorem 8.16 *Let ξ be an uncertain set whose membership function μ exists. Then ξ is (i) nonempty if and only if*

$$\sup_{x\in\Re}\mu(x)=1, \tag{8.92}$$

(ii) empty if and only if

$$\mu(x)\equiv 0, \tag{8.93}$$

and (iii) half-empty if and only if otherwise.

Proof: Since the membership function μ exists, it follows from the second measure inversion formula that

$$\mathcal{M}\{\xi=\emptyset\}=\mathcal{M}\{\xi\subset\emptyset\}=1-\sup_{x\in\emptyset^c}\mu(x)=1-\sup_{x\in\Re}\mu(x).$$

Thus ξ is (i) nonempty if and only if $\mathcal{M}\{\xi = \emptyset\} = 0$, i.e., (8.92) holds, (ii) empty if and only if $\mathcal{M}\{\xi = \emptyset\} = 1$, i.e., (8.93) holds, and (iii) half-empty if and only if otherwise.

Exercise 8.28: Some people prefer the uncertain set whose height (i.e., the supremum of the membership function) achieves 1. When the height is below 1, they divide all its membership values by the height and obtain a "normalized" membership function. Why is this idea wrong and harmful?

Inverse Membership Function

Definition 8.9 *(Liu [87]) Let ξ be an uncertain set with membership function μ. Then the set-valued function*

$$\mu^{-1}(\alpha) = \left\{x \in \Re \;\middle|\; \mu(x) \ge \alpha\right\}, \quad \forall \alpha \in [0,1] \tag{8.94}$$

is called the inverse membership function of ξ. For each given α, the set $\mu^{-1}(\alpha)$ is also called the α-cut of μ.

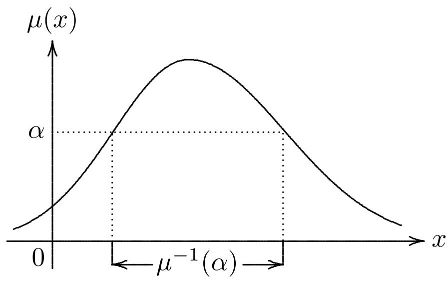


Figure 8.9: Inverse Membership Function $\mu^{-1}(\alpha)$

Remark 8.10: Let ξ be an uncertain set with inverse membership function $\mu^{-1}(\alpha)$. Then the membership function of ξ is determined by

$$\mu(x) = \sup\left\{\alpha \in [0,1] \;\middle|\; x \in \mu^{-1}(\alpha)\right\}. \tag{8.95}$$

Example 8.18: Note that an inverse membership function may take value of the empty set $\emptyset$. Let ξ be an uncertain set with membership function

$$\mu(x) = \begin{cases} 0.8, & \text{if } 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases} \tag{8.96}$$

Then its inverse membership function is

$$\mu^{-1}(\alpha) = \begin{cases} \emptyset, & \text{if } \alpha > 0.8 \\ [1,2], & \text{otherwise.} \end{cases} \tag{8.97}$$

Example 8.19: The triangular uncertain set $\xi = (a, b, c)$ has an inverse membership function

$$\mu^{-1}(\alpha) = [(1-\alpha)a + \alpha b, \alpha b + (1-\alpha)c]. \tag{8.98}$$

Example 8.20: The trapezoidal uncertain set $\xi = (a, b, c, d)$ has an inverse membership function

$$\mu^{-1}(\alpha) = [(1-\alpha)a + \alpha b, \alpha c + (1-\alpha)d]. \tag{8.99}$$

Theorem 8.17 *(Liu [87], Sufficient and Necessary Condition) A function $\mu^{-1}(\alpha)$ is an inverse membership function if and only if it is a monotone decreasing set-valued function with respect to $\alpha \in [0,1]$. That is,*

$$\mu^{-1}(\alpha) \subset \mu^{-1}(\beta), \quad \textit{if } \alpha > \beta. \tag{8.100}$$

Proof: Suppose $\mu^{-1}(\alpha)$ is an inverse membership function of some uncertain set. For any $x \in \mu^{-1}(\alpha)$, we have $\mu(x) \geq \alpha$. Since $\alpha > \beta$, we have $\mu(x) > \beta$ and then $x \in \mu^{-1}(\beta)$. Hence $\mu^{-1}(\alpha) \subset \mu^{-1}(\beta)$. Conversely, suppose $\mu^{-1}(\alpha)$ is a monotone decreasing set-valued function. Then

$$\mu(x) = \sup\left\{\alpha \in [0,1] \;\middle|\; x \in \mu^{-1}(\alpha)\right\}$$

is a membership function of some uncertain set. It is easy to verify that $\mu^{-1}(\alpha)$ is the inverse membership function of the uncertain set. The theorem is proved.

Uncertain set does not necessarily take values of its α-cut!

Please keep in mind that uncertain set does not necessarily take values of its α-cuts. In fact, an α-cut is included in the uncertain set with uncertain measure α. Conversely, the uncertain set is included in its α-cut with uncertain measure $1-\alpha$. More precisely, we have the following theorem.

Theorem 8.18 *(Liu [87]) Let ξ be an uncertain set with inverse membership function $\mu^{-1}(\alpha)$. Then for each $\alpha \in [0,1]$, we have*

$$\mathcal{M}\{\mu^{-1}(\alpha) \subset \xi\} \geq \alpha, \tag{8.101}$$

$$\mathcal{M}\{\xi \subset \mu^{-1}(\alpha)\} \geq 1-\alpha. \tag{8.102}$$

Proof: For each $x \in \mu^{-1}(\alpha)$, we have $\mu(x) \geq \alpha$. It follows from the first measure inversion formula that

$$\mathcal{M}\{\mu^{-1}(\alpha) \subset \xi\} = \inf_{x \in \mu^{-1}(\alpha)} \mu(x) \geq \alpha.$$

For each $x \notin \mu^{-1}(\alpha)$, we have $\mu(x) < \alpha$. It follows from the second measure inversion formula that

$$\mathcal{M}\{\xi \subset \mu^{-1}(\alpha)\} = 1 - \sup_{x \notin \mu^{-1}(\alpha)} \mu(x) \geq 1-\alpha.$$

Regular Membership Function

Definition 8.10 *(Liu [87]) A membership function μ of an uncertain set is said to be regular if there exists a point x_0 such that $\mu(x_0) = 1$ and $\mu(x)$ is unimodal about the mode x_0. That is, $\mu(x)$ is increasing on $(-\infty, x_0]$ and decreasing on $[x_0, +\infty)$.*

If μ is a regular membership function, then $\mu^{-1}(\alpha)$ is an interval for each α. In this case, the function

$$\mu_l^{-1}(\alpha) = \inf \mu^{-1}(\alpha) \tag{8.103}$$

is called the *left inverse membership function*, and the function

$$\mu_r^{-1}(\alpha) = \sup \mu^{-1}(\alpha) \tag{8.104}$$

is called the *right inverse membership function.* It is clear that the left inverse membership function $\mu_l^{-1}(\alpha)$ is increasing, and the right inverse membership function $\mu_r^{-1}(\alpha)$ is decreasing with respect to α.

Conversely, suppose an uncertain set ξ has a left inverse membership function $\mu_l^{-1}(\alpha)$ and right inverse membership function $\mu_r^{-1}(\alpha)$. Then the membership function μ is determined by

$$\mu(x) = \begin{cases} 0, & \text{if } x \le \mu_l^{-1}(0) \\ \alpha, & \text{if } \mu_l^{-1}(0) \le x \le \mu_l^{-1}(1) \text{ and } \mu_l^{-1}(\alpha) = x \\ 1, & \text{if } \mu_l^{-1}(1) \le x \le \mu_r^{-1}(1) \\ \beta, & \text{if } \mu_r^{-1}(1) \le x \le \mu_r^{-1}(0) \text{ and } \mu_r^{-1}(\beta) = x \\ 0, & \text{if } x \ge \mu_r^{-1}(0). \end{cases} \tag{8.105}$$

Note that the values of α and β may not be unique. In this case, we will take the maximum values.

8.3 Independence

Note that an uncertain set is a measurable function from an uncertainty space to a collection of sets of real numbers. The independence of two functions means that knowing the value of one does not change our estimation of the value of another. Two uncertain sets meet this condition if they are defined on different uncertainty spaces. For example, let $\xi_1(\gamma_1)$ and $\xi_2(\gamma_2)$ be uncertain sets on the uncertainty spaces $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$, respectively. It is clear that they are also uncertain sets on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$. Then for any Borel sets B_1 and

B_2 of real numbers, we have

$$\begin{aligned}
&\mathcal{M}\{(\xi_1 \subset B_1) \cap (\xi_2 \subset B_2)\} \\
&= \mathcal{M}\{(\gamma_1, \gamma_2) \mid \xi_1(\gamma_1) \subset B_1, \, \xi_2(\gamma_2) \subset B_2\} \\
&= \mathcal{M}\{(\gamma_1 \mid \xi_1(\gamma_1) \subset B_1) \times (\gamma_2 \mid \xi_2(\gamma_2) \subset B_2)\} \\
&= \mathcal{M}_1\{\gamma_1 \mid \xi_1(\gamma_1) \subset B_1\} \wedge \mathcal{M}_2\{\gamma_2 \mid \xi_2(\gamma_2) \subset B_2\} \\
&= \mathcal{M}\{\xi_1 \subset B_1\} \wedge \mathcal{M}\{\xi_2 \subset B_2\}.
\end{aligned}$$

That is

$$\mathcal{M}\{(\xi_1 \subset B_1) \cap (\xi_2 \subset B_2)\} = \mathcal{M}\{\xi_1 \subset B_1\} \wedge \mathcal{M}\{\xi_2 \subset B_2\}. \tag{8.106}$$

Similarly, we may verify the following seven equations:

$$\mathcal{M}\{(\xi_1^c \subset B_1) \cap (\xi_2 \subset B_2)\} = \mathcal{M}\{\xi_1^c \subset B_1\} \wedge \mathcal{M}\{\xi_2 \subset B_2\}, \tag{8.107}$$

$$\mathcal{M}\{(\xi_1 \subset B_1) \cap (\xi_2^c \subset B_2)\} = \mathcal{M}\{\xi_1 \subset B_1\} \wedge \mathcal{M}\{\xi_2^c \subset B_2\}, \tag{8.108}$$

$$\mathcal{M}\{(\xi_1^c \subset B_1) \cap (\xi_2^c \subset B_2)\} = \mathcal{M}\{\xi_1^c \subset B_1\} \wedge \mathcal{M}\{\xi_2^c \subset B_2\}, \tag{8.109}$$

$$\mathcal{M}\{(\xi_1 \subset B_1) \cup (\xi_2 \subset B_2)\} = \mathcal{M}\{\xi_1 \subset B_1\} \vee \mathcal{M}\{\xi_2 \subset B_2\}, \tag{8.110}$$

$$\mathcal{M}\{(\xi_1^c \subset B_1) \cup (\xi_2 \subset B_2)\} = \mathcal{M}\{\xi_1^c \subset B_1\} \vee \mathcal{M}\{\xi_2 \subset B_2\}, \tag{8.111}$$

$$\mathcal{M}\{(\xi_1 \subset B_1) \cup (\xi_2^c \subset B_2)\} = \mathcal{M}\{\xi_1 \subset B_1\} \vee \mathcal{M}\{\xi_2^c \subset B_2\}, \tag{8.112}$$

$$\mathcal{M}\{(\xi_1^c \subset B_1) \cup (\xi_2^c \subset B_2)\} = \mathcal{M}\{\xi_1^c \subset B_1\} \vee \mathcal{M}\{\xi_2^c \subset B_2\}. \tag{8.113}$$

Thus we say two uncertain sets are independent if the above eight equations hold. Generally, we may define independence in the following form.

Definition 8.11 *(Liu [90]) The uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i^* \subset B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i^* \subset B_i\} \tag{8.114}$$

and

$$\mathcal{M}\left\{\bigcup_{i=1}^{n} (\xi_i^* \subset B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i^* \subset B_i\} \tag{8.115}$$

where ξ_i^* are arbitrarily chosen from $\{\xi_i, \xi_i^c\}$, $i = 1, 2, \cdots, n$, respectively.

Remark 8.11: Note that (8.114) and (8.115) represent 2^{n+1} equations. For example, when $n = 2$, they represent the 8 equations from (8.106) to (8.113).

Exercise 8.29: Show that a crisp set of real numbers (a special uncertain set) is always independent of any uncertain set.

Exercise 8.30: Let ξ be an uncertain set. Are ξ and ξ^c independent? Please justify your answer.

Theorem 8.19 *(Liu [90]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain sets, and let ξ_i^* be arbitrarily chosen uncertain sets from $\{\xi_i, \xi_i^c\}$, $i = 1, 2, \cdots, n$, respectively. Then $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if $\xi_1^*, \xi_2^*, \cdots, \xi_n^*$ are independent.*

Proof: Let ξ_i^{**} be arbitrarily chosen uncertain sets from $\{\xi_i^*, \xi_i^{*c}\}$, $i = 1, 2, \cdots, n$, respectively. Then $\xi_1^*, \xi_2^*, \cdots, \xi_n^*$ and $\xi_1^{**}, \xi_2^{**}, \cdots, \xi_n^{**}$ represent the same 2^n combinations. This fact implies that (8.114) and (8.115) are equivalent to

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i^{**} \subset B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{\xi_i^{**} \subset B_i\right\}, \tag{8.116}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^{**} \subset B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\xi_i^{**} \subset B_i\right\}. \tag{8.117}$$

Hence $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if $\xi_1^*, \xi_2^*, \cdots, \xi_n^*$ are independent.

Exercise 8.31: Show that the following four statements are equivalent: (i) ξ_1 and ξ_2 are independent; (ii) ξ_1^c and ξ_2 are independent; (iii) ξ_1 and ξ_2^c are independent; and (iv) ξ_1^c and ξ_2^c are independent.

Theorem 8.20 *(Liu [90]) The uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(B_i \subset \xi_i^*)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{B_i \subset \xi_i^*\right\} \tag{8.118}$$

and

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(B_i \subset \xi_i^*)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{B_i \subset \xi_i^*\right\} \tag{8.119}$$

where ξ_i^* are arbitrarily chosen from $\{\xi_i, \xi_i^c\}$, $i = 1, 2, \cdots, n$, respectively.

Proof: Since $\{B_i \subset \xi_i^*\} = \{\xi_i^{*c} \subset B_i^c\}$ for $i = 1, 2, \cdots, n$, we immediately have

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(B_i \subset \xi_i^*)\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i^{*c} \subset B_i^c)\right\}, \tag{8.120}$$

$$\bigwedge_{i=1}^{n} \mathcal{M}\left\{B_i \subset \xi_i^*\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i^{*c} \subset B_i^c\}, \tag{8.121}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(B_i \subset \xi_i^*)\right\} = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^{*c} \subset B_i^c)\right\}, \tag{8.122}$$

$$\bigvee_{i=1}^{n} \mathcal{M}\{B_i \subset \xi_i^*\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i^{*c} \subset B_i^c\}. \tag{8.123}$$

It follows from (8.120), (8.121), (8.122) and (8.123) that (8.118) and (8.119) are valid if and only if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i^{*c} \subset B_i^c)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i^{*c} \subset B_i^c\}, \tag{8.124}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{n} (\xi_i^{*c} \subset B_i^c)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i^{*c} \subset B_i^c\}. \tag{8.125}$$

The above two equations are also equivalent to the independence of the uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$. The theorem is thus proved.

8.4 Set Operational Law

This section will discuss the union, intersection and complement of uncertain sets via membership functions.

Union of Uncertain Sets

Theorem 8.21 *(Liu [87]) Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then their union $\xi \cup \eta$ has a membership function*

$$\lambda(x) = \mu(x) \vee \nu(x). \tag{8.126}$$

Proof: In order to prove $\mu \vee \nu$ is the membership function of $\xi \cup \eta$, we must verify the two measure inversion formulas. Let B be any Borel set of real numbers, and write

$$\beta = \inf_{x \in B} \mu(x) \vee \nu(x).$$

Then $B \subset \mu^{-1}(\beta) \cup \nu^{-1}(\beta)$. By the independence of ξ and η, we have

$$\begin{aligned}
\mathcal{M}\{B \subset (\xi \cup \eta)\} &\geq \mathcal{M}\{(\mu^{-1}(\beta) \cup \nu^{-1}(\beta)) \subset (\xi \cup \eta)\} \\
&\geq \mathcal{M}\{(\mu^{-1}(\beta) \subset \xi) \cap (\nu^{-1}(\beta) \subset \eta)\} \\
&= \mathcal{M}\{\mu^{-1}(\beta) \subset \xi\} \wedge \mathcal{M}\{\nu^{-1}(\beta) \subset \eta\} \\
&\geq \beta \wedge \beta = \beta.
\end{aligned}$$

Thus

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} \geq \inf_{x \in B} \mu(x) \vee \nu(x). \tag{8.127}$$

On the other hand, for any $x \in B$, we have

$$\begin{aligned}
\mathcal{M}\{B \subset (\xi \cup \eta)\} &\leq \mathcal{M}\{x \in (\xi \cup \eta)\} = \mathcal{M}\{(x \in \xi) \cup (x \in \eta)\} \\
&= \mathcal{M}\{x \in \xi\} \vee \mathcal{M}\{x \in \eta\} = \mu(x) \vee \nu(x).
\end{aligned}$$

Thus

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} \le \inf_{x \in B} \mu(x) \vee \nu(x). \tag{8.128}$$

It follows from (8.127) and (8.128) that

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} = \inf_{x \in B} \mu(x) \vee \nu(x). \tag{8.129}$$

The first measure inversion formula is verified. Next we prove the second measure inversion formula. By the independence of ξ and η, we have

$$\begin{aligned}\mathcal{M}\{(\xi \cup \eta) \subset B\} &= \mathcal{M}\{(\xi \subset B) \cap (\eta \subset B)\} = \mathcal{M}\{\xi \subset B\} \wedge \mathcal{M}\{\eta \subset B\} \\ &= \left(1 - \sup_{x \in B^c} \mu(x)\right) \wedge \left(1 - \sup_{x \in B^c} \nu(x)\right) \\ &= 1 - \sup_{x \in B^c} \mu(x) \vee \nu(x).\end{aligned}$$

That is,

$$\mathcal{M}\{(\xi \cup \eta) \subset B\} = 1 - \sup_{x \in B^c} \mu(x) \vee \nu(x). \tag{8.130}$$

The second measure inversion formula is verified. Therefore, the union $\xi \cup \eta$ is proved to have the membership function $\mu \vee \nu$ by the measure inversion formulas (8.129) and (8.130).

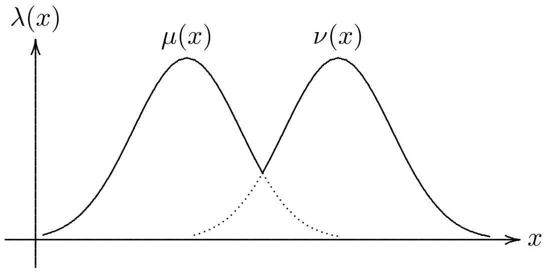


Figure 8.10: Membership Function of Union of Uncertain Sets

Example 8.21: The independence condition in Theorem 8.21 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Then

$$\xi(\gamma) = \begin{cases} [0, 1], & \text{if } \gamma = \gamma_1 \\ [0, 2], & \text{if } \gamma = \gamma_2 \end{cases}$$

is an uncertain set with membership function

$$\mu(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 0.5, & \text{if } 1 < x \le 2 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\eta(\gamma)=\begin{cases}[0,2], & \text{if } \gamma=\gamma_1\\ [0,1], & \text{if } \gamma=\gamma_2\end{cases}$$

is also an uncertain set with membership function

$$\nu(x)=\begin{cases}1, & \text{if } 0\le x\le 1\\ 0.5, & \text{if } 1<x\le 2\\ 0, & \text{otherwise.}\end{cases}$$

Note that ξ and η are not independent, and $\xi\cup\eta\equiv[0,2]$ whose membership function is

$$\lambda(x)=\begin{cases}1, & \text{if } 0\le x\le 2\\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$\lambda(x)\ne\mu(x)\vee\nu(x). \tag{8.131}$$

Therefore, the independence condition cannot be removed.

Exercise 8.32: Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent uncertain sets with membership functions $\mu_1,\mu_2,\cdots,\mu_n$, respectively. What is the membership function of $\xi_1\cup\xi_2\cup\cdots\cup\xi_n$?

Exercise 8.33: Some people suggest $\lambda(x)=\mu(x)+\nu(x)-\mu(x)\cdot\nu(x)$ and $\lambda(x)=\min\{1,\mu(x)+\nu(x)\}$ for the membership function of the union of uncertain sets. Why is this idea wrong and harmful?

Exercise 8.34: Why is $\lambda(x)=\mu(x)\vee\nu(x)$ the only option for the membership function of the union of uncertain sets?

Intersection of Uncertain Sets

Theorem 8.22 *(Liu [87]) Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then their intersection $\xi\cap\eta$ has a membership function*

$$\lambda(x)=\mu(x)\wedge\nu(x). \tag{8.132}$$

Proof: In order to prove $\mu\wedge\nu$ is the membership function of $\xi\cap\eta$, we must verify the two measure inversion formulas. Let B be any Borel set of real numbers. By the independence of ξ and η, we have

$$\begin{aligned}\mathcal{M}\{B\subset(\xi\cap\eta)\}&=\mathcal{M}\{(B\subset\xi)\cap(B\subset\eta)\}=\mathcal{M}\{B\subset\xi\}\wedge\mathcal{M}\{B\subset\eta\}\\&=\inf_{x\in B}\mu(x)\wedge\inf_{x\in B}\nu(x)=\inf_{x\in B}\mu(x)\wedge\nu(x).\end{aligned}$$

That is,

$$\mathcal{M}\{B\subset(\xi\cap\eta)\}=\inf_{x\in B}\mu(x)\wedge\nu(x). \tag{8.133}$$

The first measure inversion formula is verified. In order to prove the second measure inversion formula, we write

$$\beta = \sup_{x \in B^c} \mu(x) \wedge \nu(x).$$

Then for any given number $\varepsilon > 0$, we have $\mu^{-1}(\beta+\varepsilon) \cap \nu^{-1}(\beta+\varepsilon) \subset B$. By the independence of ξ and η, we obtain

$$\begin{aligned}
\mathcal{M}\{(\xi \cap \eta) \subset B\} &\geq \mathcal{M}\{(\xi \cap \eta) \subset (\mu^{-1}(\beta+\varepsilon) \cap \nu^{-1}(\beta+\varepsilon))\} \\
&\geq \mathcal{M}\{(\xi \subset \mu^{-1}(\beta+\varepsilon)) \cap (\eta \subset \nu^{-1}(\beta+\varepsilon))\} \\
&= \mathcal{M}\{\xi \subset \mu^{-1}(\beta+\varepsilon)\} \wedge \mathcal{M}\{\eta \subset \nu^{-1}(\beta+\varepsilon)\} \\
&\geq (1-\beta-\varepsilon) \wedge (1-\beta-\varepsilon) = 1-\beta-\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\mathcal{M}\{(\xi \cap \eta) \subset B\} \geq 1 - \sup_{x \in B^c} \mu(x) \wedge \nu(x). \tag{8.134}$$

On the other hand, for any $x \in B^c$, we have

$$\begin{aligned}
\mathcal{M}\{(\xi \cap \eta) \subset B\} &\leq \mathcal{M}\{x \notin (\xi \cap \eta)\} = \mathcal{M}\{(x \notin \xi) \cup (x \notin \eta)\} \\
&= \mathcal{M}\{x \notin \xi\} \vee \mathcal{M}\{x \notin \eta\} = (1-\mu(x)) \vee (1-\nu(x)) \\
&= 1 - \mu(x) \wedge \nu(x).
\end{aligned}$$

Thus

$$\mathcal{M}\{(\xi \cap \eta) \subset B\} \leq 1 - \sup_{x \in B^c} \mu(x) \wedge \nu(x). \tag{8.135}$$

It follows from (8.134) and (8.135) that

$$\mathcal{M}\{(\xi \cap \eta) \subset B\} = 1 - \sup_{x \in B^c} \mu(x) \wedge (x). \tag{8.136}$$

The second measure inversion formula is verified. Therefore, the intersection $\xi \cap \eta$ is proved to have the membership function $\mu \wedge \nu$ by the measure inversion formulas (8.133) and (8.136).

Example 8.22: The independence condition in Theorem 8.22 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Then

$$\xi(\gamma) = \begin{cases} [0,1], & \text{if } \gamma = \gamma_1 \\ [0,2], & \text{if } \gamma = \gamma_2 \end{cases}$$

is an uncertain set with membership function

$$\mu(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0.5, & \text{if } 1 < x \leq 2 \\ 0, & \text{otherwise,} \end{cases}$$

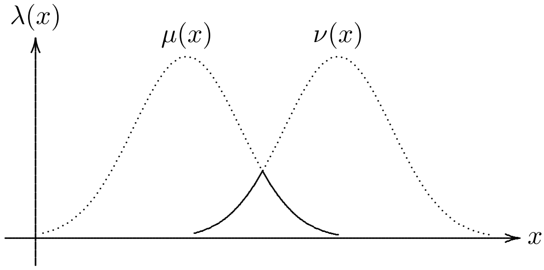


Figure 8.11: Membership Function of Intersection of Uncertain Sets

and

$$\eta(\gamma) = \begin{cases} [0,2], & \text{if } \gamma = \gamma_1 \\ [0,1], & \text{if } \gamma = \gamma_2 \end{cases}$$

is also an uncertain set with membership function

$$\nu(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 0.5, & \text{if } 1 < x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Note that ξ and η are not independent, and $\xi \cap \eta \equiv [0,1]$ whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\lambda(x) \neq \mu(x) \wedge \nu(x). \tag{8.137}$$

Therefore, the independence condition cannot be removed.

Exercise 8.35: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively. What is the membership function of $\xi_1 \cap \xi_2 \cap \cdots \cap \xi_n$?

Exercise 8.36: Some people suggest $\lambda(x) = \max\{0, \mu(x) + \nu(x) - 1\}$ and $\lambda(x) = \mu(x) \cdot \nu(x)$ for the membership function of the intersection of uncertain sets. Why is this idea wrong and harmful?

Exercise 8.37: Why is $\lambda(x) = \mu(x) \wedge \nu(x)$ the only option for the membership function of the intersection of uncertain sets?

Complement of Uncertain Set

Theorem 8.23 *(Liu [87]) Let ξ be an uncertain set with membership function μ. Then its complement ξ^c has a membership function*

$$\lambda(x) = 1 - \mu(x). \tag{8.138}$$

Proof: In order to prove $1-\mu$ is the membership function of ξ^c, we must verify the two measure inversion formulas. Let B be a Borel set of real numbers. It follows from the definition of membership function that

$$\mathcal{M}\{B \subset \xi^c\} = \mathcal{M}\{\xi \subset B^c\} = 1 - \sup_{x\in(B^c)^c} \mu(x) = \inf_{x\in B}(1-\mu(x)),$$

$$\mathcal{M}\{\xi^c \subset B\} = \mathcal{M}\{B^c \subset \xi\} = \inf_{x\in B^c} \mu(x) = 1 - \sup_{x\in B^c}(1-\mu(x)).$$

Thus ξ^c has the membership function $1-\mu$.

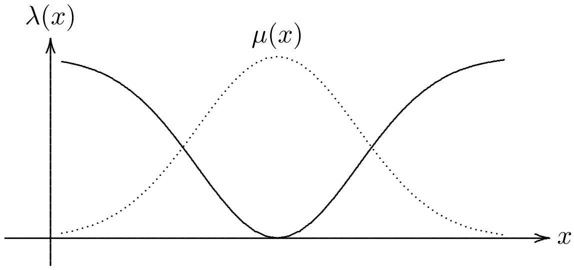


Figure 8.12: Membership Function of Complement of Uncertain Set

Exercise 8.38: Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then the set difference of ξ and η, denoted by $\xi \setminus \eta$, is the set of all elements that are members of ξ but not members of η. That is,

$$\xi \setminus \eta = \xi \cap \eta^c. \tag{8.139}$$

Show that $\xi \setminus \eta$ has a membership function

$$\lambda(x) = \mu(x) \wedge (1-\nu(x)). \tag{8.140}$$

Exercise 8.39: Let ξ be an uncertain set with membership function $\mu(x)$. Theorem 8.23 tells us that ξ^c has a membership function $1-\mu(x)$. (i) It is known that $\xi \cup \xi^c \equiv \Re$ whose membership function is $\lambda(x) \equiv 1$, and

$$\lambda(x) \neq \mu(x) \vee (1-\mu(x)). \tag{8.141}$$

Why is Theorem 8.21 not applicable to the union of ξ and ξ^c? (ii) It is known that $\xi \cap \xi^c \equiv \emptyset$ whose membership function is $\lambda(x) \equiv 0$, and

$$\lambda(x) \neq \mu(x) \wedge (1-\mu(x)). \tag{8.142}$$

Why is Theorem 8.22 not applicable to the intersection of ξ and ξ^c?

8.5 Arithmetic Operational Law

This section will present an arithmetic operational law of independent uncertain sets, including addition, subtraction, multiplication and division.

Arithmetic Operational Law via Inverse Membership Functions

Theorem 8.24 *(Liu [87]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with inverse membership functions $\mu_1^{-1}, \mu_2^{-1}, \cdots, \mu_n^{-1}$, respectively, and let f be a measurable function. Then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{8.143}$$

has an inverse membership function,

$$\lambda^{-1}(\alpha) = f(\mu_1^{-1}(\alpha), \mu_2^{-1}(\alpha), \cdots, \mu_n^{-1}(\alpha)). \tag{8.144}$$

Proof: For simplicity, we only prove the case $n = 2$. Let B be any Borel set of real numbers, and write

$$\beta = \inf_{x \in B} \lambda(x).$$

Then $B \subset \lambda^{-1}(\beta)$. Since $\lambda^{-1}(\beta) = f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta))$, by the independence of ξ_1 and ξ_2, we have

$$\begin{aligned}
\mathcal{M}\{B \subset \xi\} &\geq \mathcal{M}\{\lambda^{-1}(\beta) \subset \xi\} = \mathcal{M}\{f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta)) \subset \xi\} \\
&\geq \mathcal{M}\{(\mu_1^{-1}(\beta) \subset \xi_1) \cap (\mu_2^{-1}(\beta) \subset \xi_2)\} \\
&= \mathcal{M}\{\mu_1^{-1}(\beta) \subset \xi_1\} \wedge \mathcal{M}\{\mu_2^{-1}(\beta) \subset \xi_2\} \\
&\geq \beta \wedge \beta = \beta.
\end{aligned}$$

Thus

$$\mathcal{M}\{B \subset \xi\} \geq \inf_{x \in B} \lambda(x). \tag{8.145}$$

On the other hand, for any given number $\varepsilon > 0$, we have $B \not\subset \lambda^{-1}(\beta + \varepsilon)$. Since $\lambda^{-1}(\beta + \varepsilon) = f(\mu_1^{-1}(\beta + \varepsilon), \mu_2^{-1}(\beta + \varepsilon))$, we obtain

$$\begin{aligned}
\mathcal{M}\{B \not\subset \xi\} &\geq \mathcal{M}\{\xi \subset \lambda^{-1}(\beta + \varepsilon)\} = \mathcal{M}\{\xi \subset f(\mu_1^{-1}(\beta + \varepsilon), \mu_2^{-1}(\beta + \varepsilon))\} \\
&\geq \mathcal{M}\{(\xi_1 \subset \mu_1^{-1}(\beta + \varepsilon)) \cap (\xi_2 \subset \mu_2^{-1}(\beta + \varepsilon))\} \\
&= \mathcal{M}\{\xi_1 \subset \mu_1^{-1}(\beta + \varepsilon)\} \wedge \mathcal{M}\{\xi_2 \subset \mu_2^{-1}(\beta + \varepsilon)\} \\
&\geq (1 - \beta - \varepsilon) \wedge (1 - \beta - \varepsilon) = 1 - \beta - \varepsilon
\end{aligned}$$

and then

$$\mathcal{M}\{B \subset \xi\} = 1 - \mathcal{M}\{B \not\subset \xi\} \leq \beta + \varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\mathcal{M}\{B \subset \xi\} \leq \beta = \inf_{x \in B} \lambda(x). \tag{8.146}$$

It follows from (8.145) and (8.146) that

$$\mathcal{M}\{B \subset \xi\} = \inf_{x \in B} \lambda(x). \tag{8.147}$$

The first measure inversion formula is verified. In order to prove the second measure inversion formula, we write

$$\beta = \sup_{x \in B^c} \lambda(x).$$

Then for any given number $\varepsilon > 0$, we have $\lambda^{-1}(\beta+\varepsilon) \subset B$. Please note that $\lambda^{-1}(\beta+\varepsilon) = f(\mu_1^{-1}(\beta+\varepsilon), \mu_2^{-1}(\beta+\varepsilon))$. By the independence of ξ_1 and ξ_2, we obtain

$$\begin{aligned}
\mathcal{M}\{\xi \subset B\} &\geq \mathcal{M}\{\xi \subset \lambda^{-1}(\beta+\varepsilon)\} = \mathcal{M}\{\xi \subset f(\mu_1^{-1}(\beta+\varepsilon), \mu_2^{-1}(\beta+\varepsilon))\} \\
&\geq \mathcal{M}\{(\xi_1 \subset \mu_1^{-1}(\beta+\varepsilon)) \cap (\xi_2 \subset \mu_2^{-1}(\beta+\varepsilon))\} \\
&= \mathcal{M}\{\xi_1 \subset \mu_1^{-1}(\beta+\varepsilon)\} \wedge \mathcal{M}\{\xi_2 \subset \mu_2^{-1}(\beta+\varepsilon)\} \\
&\geq (1-\beta-\varepsilon) \wedge (1-\beta-\varepsilon) = 1-\beta-\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\mathcal{M}\{\xi \subset B\} \geq 1 - \sup_{x \in B^c} \lambda(x). \tag{8.148}$$

On the other hand, for any given number $\varepsilon > 0$, we have $\lambda^{-1}(\beta-\varepsilon) \not\subset B$. Since $\lambda^{-1}(\beta-\varepsilon) = f(\mu_1^{-1}(\beta-\varepsilon), \mu_2^{-1}(\beta-\varepsilon))$, we obtain

$$\begin{aligned}
\mathcal{M}\{\xi \not\subset B\} &\geq \mathcal{M}\{\lambda^{-1}(\beta-\varepsilon) \subset \xi\} = \mathcal{M}\{f(\mu_1^{-1}(\beta-\varepsilon), \mu_2^{-1}(\beta-\varepsilon)) \subset \xi\} \\
&\geq \mathcal{M}\{(\mu_1^{-1}(\beta-\varepsilon) \subset \xi_1) \cap (\mu_2^{-1}(\beta-\varepsilon) \subset \xi_2)\} \\
&= \mathcal{M}\{\mu_1^{-1}(\beta-\varepsilon) \subset \xi_1\} \wedge \mathcal{M}\{\mu_2^{-1}(\beta-\varepsilon) \subset \xi_2\} \\
&\geq (\beta-\varepsilon) \wedge (\beta-\varepsilon) = \beta-\varepsilon
\end{aligned}$$

and then

$$\mathcal{M}\{\xi \subset B\} = 1 - \mathcal{M}\{\xi \not\subset B\} \leq 1-\beta+\varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\mathcal{M}\{\xi \subset B\} \leq 1-\beta = 1 - \sup_{x \in B^c} \lambda(x). \tag{8.149}$$

It follows from (8.148) and (8.149) that

$$\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x \in B^c} \lambda(x). \tag{8.150}$$

The second measure inversion formula is verified. Therefore, ξ is proved to have the membership function λ by the measure inversion formulas (8.147) and (8.150).

Example 8.23: Let $\xi = (a_1, a_2, a_3)$ and $\eta = (b_1, b_2, b_3)$ be two independent triangular uncertain sets. At first, ξ has an inverse membership function,

$$\mu^{-1}(\alpha) = [(1-\alpha)a_1 + \alpha a_2, \alpha a_2 + (1-\alpha)a_3], \tag{8.151}$$

and η has an inverse membership function,

$$\nu^{-1}(\alpha) = [(1-\alpha)b_1 + \alpha b_2, \alpha b_2 + (1-\alpha)b_3]. \tag{8.152}$$

It follows from the operational law that the sum $\xi + \eta$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = [(1-\alpha)(a_1+b_1)+\alpha(a_2+b_2), \alpha(a_2+b_2)+(1-\alpha)(a_3+b_3)]. \tag{8.153}$$

In other words, the sum $\xi + \eta$ is also a triangular uncertain set, and

$$\xi + \eta = (a_1 + b_1, a_2 + b_2, a_3 + b_3). \tag{8.154}$$

Example 8.24: Let $\xi = (a_1, a_2, a_3)$ and $\eta = (b_1, b_2, b_3)$ be two independent triangular uncertain sets. It follows from the operational law that the difference $\xi - \eta$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = [(1-\alpha)(a_1-b_3)+\alpha(a_2-b_2), \alpha(a_2-b_2)+(1-\alpha)(a_3-b_1)]. \tag{8.155}$$

In other words, the difference $\xi - \eta$ is also a triangular uncertain set, and

$$\xi - \eta = (a_1 - b_3, a_2 - b_2, a_3 - b_1). \tag{8.156}$$

Example 8.25: Let $\xi = (a_1, a_2, a_3)$ be a triangular uncertain set, and k a real number. When $k \geq 0$, the product $k \cdot \xi$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = [(1-\alpha)(ka_1) + \alpha(ka_2), \alpha(ka_2) + (1-\alpha)(ka_3)]. \tag{8.157}$$

That is, the product $k \cdot \xi$ is a triangular uncertain set (ka_1, ka_2, ka_3). When $k < 0$, the product $k \cdot \xi$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = [(1-\alpha)(ka_3) + \alpha(ka_2), \alpha(ka_2) + (1-\alpha)(ka_1)]. \tag{8.158}$$

That is, the product $k \cdot \xi$ is a triangular uncertain set (ka_3, ka_2, ka_1). In summary, we have

$$k \cdot \xi = \begin{cases} (ka_1, ka_2, ka_3), & \text{if } k \geq 0 \\ (ka_3, ka_2, ka_1), & \text{if } k < 0. \end{cases} \tag{8.159}$$

Exercise 8.40: Show that the product of triangular uncertain sets is no longer a triangular one even they are independent and positive.

Exercise 8.41: Let $\xi = (a_1, a_2, a_3, a_4)$ and $\eta = (b_1, b_2, b_3, b_4)$ be two independent trapezoidal uncertain sets, and k a real number. Show that

$$\xi + \eta = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4), \tag{8.160}$$

$$\xi - \eta = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1), \tag{8.161}$$

$$k \cdot \xi = \begin{cases} (ka_1, ka_2, ka_3, ka_4), & \text{if } k \geq 0 \\ (ka_4, ka_3, ka_2, ka_1), & \text{if } k < 0. \end{cases} \tag{8.162}$$

Example 8.26: The independence condition in Theorem 8.24 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then

$$\xi_1(\gamma) = [-\gamma, \gamma] \tag{8.163}$$

is a triangular uncertain set $(-1, 0, 1)$ with inverse membership function

$$\mu_1^{-1}(\alpha) = [\alpha - 1, 1 - \alpha], \tag{8.164}$$

and

$$\xi_2(\gamma) = [\gamma - 1, 1 - \gamma] \tag{8.165}$$

is also a triangular uncertain set $(-1, 0, 1)$ with inverse membership function

$$\mu_2^{-1}(\alpha) = [\alpha - 1, 1 - \alpha]. \tag{8.166}$$

Note that ξ_1 and ξ_2 are not independent, and $\xi_1 + \xi_2 \equiv [-1, 1]$ whose inverse membership function is

$$\lambda^{-1}(\alpha) = [-1, 1]. \tag{8.167}$$

Thus

$$\lambda^{-1}(\alpha) \neq \mu_1^{-1}(\alpha) + \mu_2^{-1}(\alpha). \tag{8.168}$$

Therefore, the independence condition cannot be removed.

Monotone Function of Regular Uncertain Sets

In practice, it is usually required to deal with monotone functions of regular uncertain sets. In this case, we have the following shortcut.

Theorem 8.25 *(Liu [87]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with regular membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{8.169}$$

has a regular membership function, and

$$\lambda_l^{-1}(\alpha) = f(\mu_{1l}^{-1}(\alpha), \cdots, \mu_{ml}^{-1}(\alpha), \mu_{m+1,r}^{-1}(\alpha), \cdots, \mu_{nr}^{-1}(\alpha)), \tag{8.170}$$

$$\lambda_r^{-1}(\alpha) = f(\mu_{1r}^{-1}(\alpha), \cdots, \mu_{mr}^{-1}(\alpha), \mu_{m+1,l}^{-1}(\alpha), \cdots, \mu_{nl}^{-1}(\alpha)), \tag{8.171}$$

where $\lambda_l^{-1}, \mu_{1l}^{-1}, \mu_{2l}^{-1}, \cdots, \mu_{nl}^{-1}$ *are left inverse membership functions, and* λ_r^{-1}, $\mu_{1r}^{-1}, \mu_{2r}^{-1}, \cdots, \mu_{nr}^{-1}$ *are right inverse membership functions of* $\xi, \xi_1, \xi_2, \cdots, \xi_n$, *respectively.*

Proof: Note that $\mu_1^{-1}(\alpha), \mu_2^{-1}(\alpha), \cdots, \mu_n^{-1}(\alpha)$ are intervals for each α. Since $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, the value

$$\lambda^{-1}(\alpha) = f(\mu_1^{-1}(\alpha), \cdots, \mu_m^{-1}(\alpha), \mu_{m+1}^{-1}(\alpha), \cdots, \mu_n^{-1}(\alpha))$$

is also an interval. Thus ξ has a regular membership function, and its left and right inverse membership functions are determined by (8.170) and (8.171), respectively.

Exercise 8.42: Let ξ and η be independent uncertain sets with left inverse membership functions μ_l^{-1} and ν_l^{-1} and right inverse membership functions μ_r^{-1} and ν_r^{-1}, respectively. Show that the sum $\xi + \eta$ has left and right inverse membership functions,

$$\lambda_l^{-1}(\alpha) = \mu_l^{-1}(\alpha) + \nu_l^{-1}(\alpha), \tag{8.172}$$

$$\lambda_r^{-1}(\alpha) = \mu_r^{-1}(\alpha) + \nu_r^{-1}(\alpha). \tag{8.173}$$

Exercise 8.43: Let ξ and η be independent uncertain sets with left inverse membership functions μ_l^{-1} and ν_l^{-1} and right inverse membership functions μ_r^{-1} and ν_r^{-1}, respectively. Show that the difference $\xi - \eta$ has left and right inverse membership functions,

$$\lambda_l^{-1}(\alpha) = \mu_l^{-1}(\alpha) - \nu_r^{-1}(\alpha), \tag{8.174}$$

$$\lambda_r^{-1}(\alpha) = \mu_r^{-1}(\alpha) - \nu_l^{-1}(\alpha). \tag{8.175}$$

Exercise 8.44: Let ξ and η be independent and positive uncertain sets with left inverse membership functions μ_l^{-1} and ν_l^{-1} and right inverse membership functions μ_r^{-1} and ν_r^{-1}, respectively. Show that

$$\frac{\xi}{\xi + \eta} \tag{8.176}$$

has left and right inverse membership functions,

$$\lambda_l^{-1}(\alpha) = \frac{\mu_l^{-1}(\alpha)}{\mu_l^{-1}(\alpha) + \nu_r^{-1}(\alpha)}, \tag{8.177}$$

$$\lambda_r^{-1}(\alpha) = \frac{\mu_r^{-1}(\alpha)}{\mu_r^{-1}(\alpha) + \nu_l^{-1}(\alpha)}. \tag{8.178}$$

Arithmetic Operational Law via Membership Functions

Theorem 8.26 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with membership functions $\mu_1(x), \mu_2(x), \cdots, \mu_n(x)$, respectively, and let f be a measurable function. Then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{8.179}$$

has a membership function,

$$\lambda(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \min_{1\le i\le n} \mu_i(x_i). \tag{8.180}$$

Proof: Let λ be the membership function of ξ. For any given real number x, write $\lambda(x) = \beta$. By using Theorem 8.24, we get

$$\lambda^{-1}(\beta) = f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta), \cdots, \mu_n^{-1}(\beta)).$$

Since $x \in \lambda^{-1}(\beta)$, there exist real numbers $x_i \in \mu_i^{-1}(\beta)$, $i = 1, 2, \cdots, n$ such that $f(x_1, x_2, \cdots, x_n) = x$. Noting that $\mu_i(x_i) \ge \beta$ for $i = 1, 2, \cdots, n$, we have

$$\lambda(x) = \beta \le \min_{1\le i\le n} \mu_i(x_i)$$

and then

$$\lambda(x) \le \sup_{f(x_1,x_2,\cdots,x_n)=x} \min_{1\le i\le n} \mu_i(x_i). \tag{8.181}$$

On the other hand, assume $x_1, x_2, \cdots, x_n$ are any given real numbers with $f(x_1, x_2, \cdots, x_n) = x$. Write

$$\min_{1\le i\le n} \mu_i(x_i) = \beta.$$

By using Theorem 8.24, we get

$$\lambda^{-1}(\beta) = f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta), \cdots, \mu_n^{-1}(\beta)).$$

Noting that $x_i \in \mu_i^{-1}(\beta)$ for $i = 1, 2, \cdots, n$, we have

$$x = f(x_1, x_2, \cdots, x_n) \in f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta), \cdots, \mu_n^{-1}(\beta)) = \lambda^{-1}(\beta).$$

Hence

$$\lambda(x) \ge \beta = \min_{1\le i\le n} \mu_i(x_i)$$

and then

$$\lambda(x) \ge \sup_{f(x_1,x_2,\cdots,x_n)=x} \min_{1\le i\le n} \mu_i(x_i). \tag{8.182}$$

It follows from (8.181) and (8.182) that (8.180) holds.

Remark 8.12: It is possible that the equation $f(x_1, x_2, \cdots, x_n) = x$ does not have a root for some values of x. In this case, we set $\lambda(x) = 0$.

Example 8.27: The independence condition in Theorem 8.26 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Then

$$\xi_1(\gamma) = [-\gamma, \gamma] \tag{8.183}$$

is a triangular uncertain set $(-1, 0, 1)$ with membership function

$$\mu_1(x) = \begin{cases} 1 - |x|, & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise,} \end{cases} \tag{8.184}$$

and

$$\xi_2(\gamma) = [\gamma - 1, 1 - \gamma] \tag{8.185}$$

is also a triangular uncertain set $(-1, 0, 1)$ with membership function

$$\mu_2(x) = \begin{cases} 1 - |x|, & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{8.186}$$

Note that ξ_1 and ξ_2 are not independent, and $\xi_1 + \xi_2 \equiv [-1, 1]$ whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{8.187}$$

Thus

$$\lambda(x) \ne \sup_{x_1 + x_2 = x} \mu_1(x_1) \wedge \mu_2(x_2). \tag{8.188}$$

Therefore, the independence condition cannot be removed.

Exercise 8.45: Let ξ and η be independent uncertain sets with membership functions $\mu(x)$ and $\nu(x)$, respectively. Show that $\xi + \eta$ has a membership function,

$$\lambda(x) = \sup_{y \in \Re} \mu(x - y) \wedge \nu(y). \tag{8.189}$$

Exercise 8.46: Let ξ and η be independent uncertain sets with membership functions $\mu(x)$ and $\nu(x)$, respectively. Show that $\xi - \eta$ has a membership function,

$$\lambda(x) = \sup_{y \in \Re} \mu(x + y) \wedge \nu(y). \tag{8.190}$$

8.6 Inclusion Relation

Let ξ be an uncertain set with membership function μ, and let B be a Borel set of real numbers. By using the definition of membership function, Liu

[87] presented two measure inversion formulas for calculating the uncertain measure of inclusion relation,

$$\mathcal{M}\{B \subset \xi\} = \inf_{x \in B} \mu(x), \tag{8.191}$$

$$\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x \in B^c} \mu(x). \tag{8.192}$$

Especially, for any point x, Liu [87] also gave a formula for calculating the uncertain measure of containment relation,

$$\mathcal{M}\{x \in \xi\} = \mu(x). \tag{8.193}$$

A general formula was derived by Yao [177] for calculating the uncertain measure of inclusion relation between uncertain sets.

Theorem 8.27 *(Yao [177]) Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then*

$$\mathcal{M}\{\xi \subset \eta\} = \inf_{x \in \Re} (1 - \mu(x)) \vee \nu(x). \tag{8.194}$$

Proof: Note that $\xi \cap \eta^c$ has a membership function $\lambda(x) = \mu(x) \wedge (1 - \nu(x))$. It follows from $\{\xi \subset \eta\} \equiv \{\xi \cap \eta^c = \emptyset\}$ and the second measure inversion formula that

$$\begin{aligned} \mathcal{M}\{\xi \subset \eta\} &= \mathcal{M}\{\xi \cap \eta^c = \emptyset\} \\ &= \mathcal{M}\{\xi \cap \eta^c \subset \emptyset\} \\ &= 1 - \sup_{x \in \emptyset^c} \mu(x) \wedge (1 - \nu(x)) \\ &= \inf_{x \in \Re} (1 - \mu(x)) \vee \nu(x). \end{aligned}$$

The theorem is proved.

Example 8.28: Consider two special uncertain sets $\xi = [1, 2]$ and $\eta = [0, 3]$ that are essentially crisp intervals whose membership functions are

$$\mu(x) = \begin{cases} 1, & \text{if } 1 \le x \le 2 \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu(x) = \begin{cases} 1, & \text{if } 0 \le x \le 3 \\ 0, & \text{otherwise,} \end{cases}$$

respectively. Mention that $\xi \subset \eta$ is a completely true relation since $[1, 2]$ is indeed included in $[0, 3]$. By using (8.194), we also obtain

$$\mathcal{M}\{\xi \subset \eta\} = \inf_{x \in \Re} (1 - \mu(x)) \vee \nu(x) = 1.$$

Example 8.29: Consider two special uncertain sets $\xi = [0,2]$ and $\eta = [1,3]$ that are essentially crisp intervals whose membership functions are

$$\mu(x) = \begin{cases} 1, & \text{if } 0 \le x \le 2 \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu(x) = \begin{cases} 1, & \text{if } 1 \le x \le 3 \\ 0, & \text{otherwise,} \end{cases}$$

respectively. Mention that $\xi \subset \eta$ is a completely false relation since $[0,2]$ is not a subset of $[1,3]$. By using (8.194), we also obtain

$$\mathcal{M}\{\xi \subset \eta\} = \inf_{x \in \Re} (1 - \mu(x)) \vee \nu(x) = 0.$$

Example 8.30: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.8, & \text{if } \gamma_1 \in \Lambda \ne \Gamma \\ 0.2, & \text{if } \gamma_1 \notin \Lambda \ne \emptyset. \end{cases} \tag{8.195}$$

Define two uncertain sets,

$$\xi(\gamma) = \begin{cases} [0,3], & \text{if } \gamma = \gamma_1 \text{ or } \gamma_2 \\ [1,2], & \text{if } \gamma = \gamma_3 \text{ or } \gamma_4, \end{cases} \tag{8.196}$$

$$\eta(\gamma) = \begin{cases} [0,3], & \text{if } \gamma = \gamma_1 \text{ or } \gamma_3 \\ [1,2], & \text{if } \gamma = \gamma_2 \text{ or } \gamma_4. \end{cases} \tag{8.197}$$

We may verify that ξ and η are independent, and share a common membership function,

$$\mu(x) = \begin{cases} 1, & \text{if } 1 \le x \le 2 \\ 0.8, & \text{if } 0 \le x < 1 \text{ or } 2 < x \le 3 \\ 0, & \text{otherwise.} \end{cases} \tag{8.198}$$

Note that

$$\mathcal{M}\{\xi \subset \eta\} = \mathcal{M}\{\gamma_1, \gamma_3, \gamma_4\} = 0.8. \tag{8.199}$$

By using (8.194), we also obtain

$$\mathcal{M}\{\xi \subset \eta\} = \inf_{x \in \Re} (1 - \mu(x)) \vee \mu(x) = 0.8. \tag{8.200}$$

Exercise 8.47: Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Show that if $\mu \le \nu$, then

$$\mathcal{M}\{\xi \subset \eta\} \ge 0.5. \tag{8.201}$$

Exercise 8.48: Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively, and let c be a number between 0.5 and 1. (i) Construct ξ and η such that

$$\mu \equiv \nu \quad \text{and} \quad \mathcal{M}\{\xi \subset \eta\} = c. \tag{8.202}$$

(ii) Is it possible to re-do (i) when c is below 0.5? (iii) Is it stupid to think that $\xi \subset \eta$ if and only if $\mu(x) \le \nu(x)$ for all x? (iv) Is it stupid to think that $\xi = \eta$ if and only if $\mu(x) = \nu(x)$ for all x? (Hint: Use (8.195), (8.196) and (8.197) as a reference.)

Example 8.31: The independence condition in Theorem 8.27 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then

$$\xi(\gamma) = [-\gamma, \gamma] \tag{8.203}$$

is a triangular uncertain set $(-1, 0, 1)$ with membership function

$$\mu(x) = \begin{cases} 1 - |x|, & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise,} \end{cases} \tag{8.204}$$

and

$$\eta(\gamma) = [-\gamma, \gamma] \tag{8.205}$$

is also a triangular uncertain set $(-1, 0, 1)$ with membership function

$$\nu(x) = \begin{cases} 1 - |x|, & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{8.206}$$

Note that ξ and η are not independent (in fact, they are the same one), and $\mathcal{M}\{\xi \subset \eta\} = 1$. However, by using (8.194), we obtain

$$\mathcal{M}\{\xi \subset \eta\} = \inf_{x \in \Re} (1 - \mu(x)) \vee \nu(x) = 0.5 \ne 1. \tag{8.207}$$

Thus the independence condition cannot be removed.

8.7 Expected Value

This section will introduce a concept of expected value for nonempty uncertain set (Empty set and half-empty uncertain set have no expected value).

Definition 8.12 *(Liu [81]) Let ξ be a nonempty uncertain set. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \succeq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \preceq x\}\mathrm{d}x \tag{8.208}$$

provided that at least one of the two integrals is finite.

Please note that $\xi \succeq x$ represents "ξ is imaginarily included in $[x, +\infty)$", and $\xi \preceq x$ represents "ξ is imaginarily included in $(-\infty, x]$". What are the appropriate values of $\mathcal{M}\{\xi \succeq x\}$ and $\mathcal{M}\{\xi \preceq x\}$? Unfortunately, this problem is not as simple as you think.

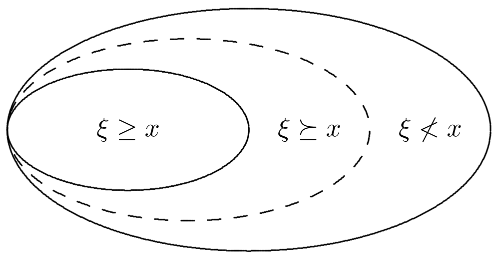


Figure 8.13: $\{\xi \geq x\} \subset \{\xi \succeq x\} \subset \{\xi \not< x\}$

It is clear that the imaginary event $\{\xi \succeq x\}$ is one between $\{\xi \geq x\}$ and $\{\xi \not< x\}$. See Figure 8.13. Intuitively, for the value of $\mathcal{M}\{\xi \succeq x\}$, it is too conservative if we take $\mathcal{M}\{\xi \geq x\}$, and it is too adventurous if we take $\mathcal{M}\{\xi \not< x\} = 1 - \mathcal{M}\{\xi < x\}$. Thus we assign $\mathcal{M}\{\xi \succeq x\}$ the middle value between $\mathcal{M}\{\xi \geq x\}$ and $1 - \mathcal{M}\{\xi < x\}$. That is,

$$\mathcal{M}\{\xi \succeq x\} = \frac{1}{2}\left(\mathcal{M}\{\xi \geq x\} + 1 - \mathcal{M}\{\xi < x\}\right). \tag{8.209}$$

Similarly, we also define

$$\mathcal{M}\{\xi \preceq x\} = \frac{1}{2}\left(\mathcal{M}\{\xi \leq x\} + 1 - \mathcal{M}\{\xi > x\}\right). \tag{8.210}$$

Example 8.32: Let $[a, b]$ be a crisp interval and assume $a > 0$ for simplicity. Then

$$\xi(\gamma) \equiv [a, b], \quad \forall \gamma \in \Gamma$$

is a special uncertain set. It follows from the definition of $\mathcal{M}\{\xi \succeq x\}$ and $\mathcal{M}\{\xi \preceq x\}$ that

$$\mathcal{M}\{\xi \succeq x\} = \begin{cases} 1, & \text{if } x \leq a \\ 0.5, & \text{if } a < x \leq b \\ 0, & \text{if } x > b, \end{cases}$$

$$\mathcal{M}\{\xi \preceq x\} \equiv 0, \quad \forall x \leq 0.$$

Thus

$$E[\xi] = \int_0^a 1\mathrm{d}x + \int_a^b 0.5\mathrm{d}x = \frac{a+b}{2}.$$

Example 8.33: In order to further illustrate the expected value operator, let us consider an uncertain set,

$$\xi = \begin{cases} [1,2] \text{ with uncertain measure } 0.6 \\ [2,3] \text{ with uncertain measure } 0.3 \\ [3,4] \text{ with uncertain measure } 0.2. \end{cases}$$

It follows from the definition of $\mathcal{M}\{\xi \succeq x\}$ and $\mathcal{M}\{\xi \preceq x\}$ that

$$\mathcal{M}\{\xi \succeq x\} = \begin{cases} 1, & \text{if } x \le 1 \\ 0.7, & \text{if } 1 < x \le 2 \\ 0.3, & \text{if } 2 < x \le 3 \\ 0.1, & \text{if } 3 < x \le 4 \\ 0, & \text{if } x > 4, \end{cases}$$

$$\mathcal{M}\{\xi \preceq x\} \equiv 0, \quad \forall x \le 0.$$

Thus

$$E[\xi] = \int_0^1 1\mathrm{d}x + \int_1^2 0.7\mathrm{d}x + \int_2^3 0.3\mathrm{d}x + \int_3^4 0.1\mathrm{d}x = 2.1.$$

How to Obtain Expected Value from Membership Function?

Let ξ be an uncertain set with membership function μ. In order to calculate its expected value via (8.208), we must determine the values of $\mathcal{M}\{\xi \succeq x\}$ and $\mathcal{M}\{\xi \preceq x\}$ from the membership function μ.

Theorem 8.28 *(Liu [83]) Let ξ be a nonempty uncertain set with membership function μ. Then for any real number x, we have*

$$\mathcal{M}\{\xi \succeq x\} = \frac{1}{2}\left(\sup_{y \ge x} \mu(y) + 1 - \sup_{y < x} \mu(y)\right), \tag{8.211}$$

$$\mathcal{M}\{\xi \preceq x\} = \frac{1}{2}\left(\sup_{y \le x} \mu(y) + 1 - \sup_{y > x} \mu(y)\right). \tag{8.212}$$

Proof: Since the uncertain set ξ has a membership function μ, the second measure inversion formula tells us that

$$\mathcal{M}\{\xi \ge x\} = 1 - \sup_{y < x} \mu(y),$$

$$\mathcal{M}\{\xi < x\} = 1 - \sup_{y \geq x} \mu(y).$$

Thus (8.211) follows from (8.209) immediately. We may also prove (8.212) similarly.

Theorem 8.29 *(Liu [83]) Let ξ be a nonempty uncertain set with membership function μ. Then*

$$E[\xi] = x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \sup_{y \geq x} \mu(y)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \sup_{y \leq x} \mu(y)\mathrm{d}x \tag{8.213}$$

where x_0 is a point such that $\mu(x_0) = 1$.

Proof: Since μ achieves 1 at x_0, it follows from Theorem 8.28 that for almost all x, we have

$$\mathcal{M}\{\xi \succeq x\} = \begin{cases} 1 - \sup\limits_{y < x} \mu(x)/2, & \text{if } x \leq x_0 \\ \sup\limits_{y \geq x} \mu(x)/2, & \text{if } x > x_0 \end{cases} \tag{8.214}$$

and

$$\mathcal{M}\{\xi \preceq x\} = \begin{cases} \sup\limits_{y \leq x} \mu(x)/2, & \text{if } x < x_0 \\ 1 - \sup\limits_{y > x} \mu(x)/2, & \text{if } x \geq x_0. \end{cases} \tag{8.215}$$

If $x_0 \geq 0$, then

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \succeq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \preceq x\}\mathrm{d}x \\ &= \int_0^{x_0} \left(1 - \sup_{y \leq x} \frac{\mu(x)}{2}\right)\mathrm{d}x + \int_{x_0}^{+\infty} \sup_{y \geq x} \frac{\mu(x)}{2}\mathrm{d}x - \int_{-\infty}^0 \sup_{y \leq x} \frac{\mu(x)}{2}\mathrm{d}x \\ &= x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \sup_{y \geq x} \mu(y)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \sup_{y \leq x} \mu(y)\mathrm{d}x. \end{aligned}$$

If $x_0 < 0$, then

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \succeq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \preceq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \sup_{y \geq x} \frac{\mu(x)}{2}\mathrm{d}x - \int_{-\infty}^{x_0} \sup_{y \leq x} \frac{\mu(x)}{2}\mathrm{d}x - \int_{x_0}^0 \left(1 - \sup_{y \geq x} \frac{\mu(x)}{2}\right)\mathrm{d}x \\ &= x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \sup_{y \geq x} \mu(y)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \sup_{y \leq x} \mu(y)\mathrm{d}x. \end{aligned}$$

The theorem is thus proved.

Theorem 8.30 *(Liu [83]) Let ξ be an uncertain set with regular membership function μ. Then*

$$E[\xi] = x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \mu(x)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \mu(x)\mathrm{d}x \tag{8.216}$$

where x_0 is a point such that $\mu(x_0) = 1$.

Proof: Since μ is increasing on $(-\infty, x_0]$ and decreasing on $[x_0, +\infty)$, for almost all $x \geq x_0$, we have

$$\sup_{y \geq x} \mu(y) = \mu(x); \tag{8.217}$$

and for almost all $x \leq x_0$, we have

$$\sup_{y \leq x} \mu(y) = \mu(x). \tag{8.218}$$

Thus the theorem follows from (8.213) immediately.

Exercise 8.49: Show that the triangular uncertain set $\xi = (a, b, c)$ has an expected value

$$E[\xi] = \frac{a + 2b + c}{4}. \tag{8.219}$$

Exercise 8.50: Show that the trapezoidal uncertain set $\xi = (a, b, c, d)$ has an expected value

$$E[\xi] = \frac{a + b + c + d}{4}. \tag{8.220}$$

Theorem 8.31 *(Liu [87]) Let ξ be a nonempty uncertain set with membership function μ. Then*

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)\right) \mathrm{d}\alpha \tag{8.221}$$

where $\inf \mu^{-1}(\alpha)$ and $\sup \mu^{-1}(\alpha)$ are the infimum and supremum of the α-cut, respectively.

Proof: Since ξ is a nonempty uncertain set and has a finite expected value, we may assume that there exists a point x_0 such that $\mu(x_0) = 1$ (perhaps after a small perturbation). It is clear that the two integrals

$$\int_{x_0}^{+\infty} \sup_{y \geq x} \mu(y)\mathrm{d}x \quad \text{and} \quad \int_0^1 (\sup \mu^{-1}(\alpha) - x_0)\mathrm{d}\alpha$$

make an identical acreage. Thus

$$\int_{x_0}^{+\infty} \sup_{y \geq x} \mu(y)\mathrm{d}x = \int_0^1 (\sup \mu^{-1}(\alpha) - x_0)\mathrm{d}\alpha = \int_0^1 \sup \mu^{-1}(\alpha)\mathrm{d}\alpha - x_0.$$

Similarly, we may prove

$$\int_{-\infty}^{x_0} \sup_{y\le x} \mu(y)\mathrm{d}x = \int_0^1 (x_0 - \inf \mu^{-1}(\alpha))\mathrm{d}\alpha = x_0 - \int_0^1 \inf \mu^{-1}(\alpha)\mathrm{d}\alpha.$$

It follows from (8.213) that

$$\begin{aligned} E[\xi] &= x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \sup_{y\ge x} \mu(y)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \sup_{y\le x} \mu(y)\mathrm{d}x \\ &= x_0 + \frac{1}{2}\left(\int_0^1 \sup \mu^{-1}(\alpha)\mathrm{d}\alpha - x_0\right) - \frac{1}{2}\left(x_0 - \int_0^1 \inf \mu^{-1}(\alpha)\mathrm{d}\alpha\right) \\ &= \frac{1}{2}\int_0^1 (\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha))\mathrm{d}\alpha. \end{aligned}$$

The theorem is thus verified.

Theorem 8.32 *(Liu [87]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with regular membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{8.222}$$

has an expected value

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\mu_l^{-1}(\alpha) + \mu_r^{-1}(\alpha)\right)\mathrm{d}\alpha \tag{8.223}$$

where $\mu_l^{-1}(\alpha)$ and $\mu_r^{-1}(\alpha)$ are determined by

$$\mu_l^{-1}(\alpha) = f(\mu_{1l}^{-1}(\alpha), \cdots, \mu_{ml}^{-1}(\alpha), \mu_{m+1,r}^{-1}(\alpha), \cdots, \mu_{nr}^{-1}(\alpha)), \tag{8.224}$$

$$\mu_r^{-1}(\alpha) = f(\mu_{1r}^{-1}(\alpha), \cdots, \mu_{mr}^{-1}(\alpha), \mu_{m+1,l}^{-1}(\alpha), \cdots, \mu_{nl}^{-1}(\alpha)). \tag{8.225}$$

Proof: It follows from Theorems 8.25 and 8.31 immediately.

Exercise 8.51: Let ξ and η be independent and nonnegative uncertain sets with regular membership functions μ and ν, respectively. Show that

$$E[\xi\eta] = \frac{1}{2}\int_0^1 \left(\mu_l^{-1}(\alpha)\nu_l^{-1}(\alpha) + \mu_r^{-1}(\alpha)\nu_r^{-1}(\alpha)\right)\mathrm{d}\alpha. \tag{8.226}$$

Exercise 8.52: Let ξ and η be independent and positive uncertain sets with regular membership functions μ and ν, respectively. Show that

$$E\left[\frac{\xi}{\eta}\right] = \frac{1}{2}\int_0^1 \left(\frac{\mu_l^{-1}(\alpha)}{\nu_r^{-1}(\alpha)} + \frac{\mu_r^{-1}(\alpha)}{\nu_l^{-1}(\alpha)}\right)\mathrm{d}\alpha. \tag{8.227}$$

Exercise 8.53: Let ξ and η be independent and positive uncertain sets with regular membership functions μ and ν, respectively. Show that

$$E\left[\frac{\xi}{\xi+\eta}\right]=\frac{1}{2}\int_0^1\left(\frac{\mu_l^{-1}(\alpha)}{\mu_l^{-1}(\alpha)+\nu_r^{-1}(\alpha)}+\frac{\mu_r^{-1}(\alpha)}{\mu_r^{-1}(\alpha)+\nu_l^{-1}(\alpha)}\right)\mathrm{d}\alpha. \quad (8.228)$$

Linearity of Expected Value Operator

Theorem 8.33 *(Liu [87]) Let ξ and η be independent uncertain sets with finite expected values. Then for any real numbers a and b, we have*

$$E[a\xi+b\eta]=aE[\xi]+bE[\eta]. \quad (8.229)$$

Proof: Denote the membership functions of ξ and η by μ and ν, respectively. Then

$$E[\xi]=\frac{1}{2}\int_0^1\left(\inf\mu^{-1}(\alpha)+\sup\mu^{-1}(\alpha)\right)\mathrm{d}\alpha,$$

$$E[\eta]=\frac{1}{2}\int_0^1\left(\inf\nu^{-1}(\alpha)+\sup\nu^{-1}(\alpha)\right)\mathrm{d}\alpha.$$

STEP 1: We first prove $E[a\xi]=aE[\xi]$. The product $a\xi$ has an inverse membership function,

$$\lambda^{-1}(\alpha)=a\mu^{-1}(\alpha).$$

It follows from Theorem 8.31 that

$$\begin{aligned}E[a\xi]&=\frac{1}{2}\int_0^1\left(\inf\lambda^{-1}(\alpha)+\sup\lambda^{-1}(\alpha)\right)\mathrm{d}\alpha\\&=\frac{a}{2}\int_0^1\left(\inf\mu^{-1}(\alpha)+\sup\mu^{-1}(\alpha)\right)\mathrm{d}\alpha=aE[\xi].\end{aligned}$$

STEP 2: We then prove $E[\xi+\eta]=E[\xi]+E[\eta]$. The sum $\xi+\eta$ has an inverse membership function,

$$\lambda^{-1}(\alpha)=\mu^{-1}(\alpha)+\nu^{-1}(\alpha).$$

It follows from Theorem 8.31 that

$$\begin{aligned}E[\xi+\eta]&=\frac{1}{2}\int_0^1\left(\inf\lambda^{-1}(\alpha)+\sup\lambda^{-1}(\alpha)\right)\mathrm{d}\alpha\\&=\frac{1}{2}\int_0^1\left(\inf\mu^{-1}(\alpha)+\sup\mu^{-1}(\alpha)\right)\mathrm{d}\alpha\\&\quad+\frac{1}{2}\int_0^1\left(\inf\nu^{-1}(\alpha)+\sup\nu^{-1}(\alpha)\right)\mathrm{d}\alpha\\&=E[\xi]+E[\eta].\end{aligned}$$

STEP 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$E[a\xi + b\eta] = E[a\xi] + E[b\eta] = aE[\xi] + bE[\eta].$$

The theorem is proved.

Example 8.34: Generally speaking, the expected value operator is not necessarily linear if the independence is not assumed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.3$, $\mathcal{M}\{\gamma_3\} = 0.2$. Define two uncertain sets as follows,

$$\xi(\gamma) = \begin{cases} [1,4], & \text{if } \gamma = \gamma_1 \\ [1,3], & \text{if } \gamma = \gamma_2 \\ [1,2], & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} [1,5], & \text{if } \gamma = \gamma_1 \\ [1,2], & \text{if } \gamma = \gamma_2 \\ [1,4], & \text{if } \gamma = \gamma_3. \end{cases}$$

Note that ξ and η are not independent, and their sum is

$$(\xi + \eta)(\gamma) = \begin{cases} [2,9], & \text{if } \gamma = \gamma_1 \\ [2,5], & \text{if } \gamma = \gamma_2 \\ [2,6], & \text{if } \gamma = \gamma_3. \end{cases}$$

It is easy to verify that $E[\xi] = 2.2$, $E[\eta] = 2.5$ and $E[\xi + \eta] = 4.75$. Thus we have

$$E[\xi + \eta] > E[\xi] + E[\eta].$$

If the uncertain sets are defined by

$$\xi(\gamma) = \begin{cases} [1,4], & \text{if } \gamma = \gamma_1 \\ [1,3], & \text{if } \gamma = \gamma_2 \\ [1,2], & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} [1,4], & \text{if } \gamma = \gamma_1 \\ [1,6], & \text{if } \gamma = \gamma_2 \\ [1,2], & \text{if } \gamma = \gamma_3, \end{cases}$$

then

$$(\xi + \eta)(\gamma) = \begin{cases} [2,8], & \text{if } \gamma = \gamma_1 \\ [2,9], & \text{if } \gamma = \gamma_2 \\ [2,4], & \text{if } \gamma = \gamma_3. \end{cases}$$

It is easy to verify that $E[\xi] = 2.2$, $E[\eta] = 2.6$ and $E[\xi + \eta] = 4.75$. Thus we have

$$E[\xi + \eta] < E[\xi] + E[\eta].$$

Therefore, the independence condition cannot be removed.

8.8 Variance

The variance of uncertain set provides a degree of the spread of the membership function around its expected value.

Definition 8.13 *(Liu [84]) Let ξ be an uncertain set with finite expected value e. Then the variance of ξ is defined by*

$$V[\xi] = E[(\xi - e)^2]. \tag{8.230}$$

This definition says that the variance is just the expected value of $(\xi - e)^2$. Since $(\xi - e)^2$ is a nonnegative uncertain set, we also have

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \succeq x\}\mathrm{d}x. \tag{8.231}$$

Please note that $(\xi - e)^2 \succeq x$ represents "$(\xi - e)^2$ is imaginarily included in $[x, +\infty)$". What is the appropriate value of $\mathcal{M}\{(\xi - e)^2 \succeq x\}$? Intuitively, it is too conservative if we take the value $\mathcal{M}\{(\xi - e)^2 \ge x\}$, and it is too adventurous if we take the value $1 - \mathcal{M}\{(\xi - e)^2 < x\}$. Thus we assign $\mathcal{M}\{(\xi - e)^2 \succeq x\}$ the middle value between them. That is,

$$\mathcal{M}\{(\xi - e)^2 \succeq x\} = \frac{1}{2}\left(\mathcal{M}\{(\xi - e)^2 \ge x\} + 1 - \mathcal{M}\{(\xi - e)^2 < x\}\right). \tag{8.232}$$

Theorem 8.34 *If ξ is an uncertain set with finite expected value, a and b are real numbers, then*

$$V[a\xi + b] = a^2 V[\xi]. \tag{8.233}$$

Proof: If ξ has an expected value e, then $a\xi + b$ has an expected value $ae + b$. It follows from the definition of variance that

$$V[a\xi + b] = E\left[(a\xi + b - ae - b)^2\right] = a^2 E[(\xi - e)^2] = a^2 V[\xi].$$

Theorem 8.35 *Let ξ be an uncertain set with expected value e. Then $V[\xi] = 0$ if and only if $\xi = \{e\}$ almost surely.*

Proof: We first assume $V[\xi] = 0$. It follows from the equation (8.231) that

$$\int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \succeq x\}\mathrm{d}x = 0$$

which implies $\mathcal{M}\{(\xi - e)^2 \succeq x\} = 0$ for any $x > 0$. Hence $\mathcal{M}\{\xi = \{e\}\} = 1$. Conversely, assume $\mathcal{M}\{\xi = \{e\}\} = 1$. Then we have $\mathcal{M}\{(\xi - e)^2 \succeq x\} = 0$ for any $x > 0$. Thus

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \succeq x\}\mathrm{d}x = 0.$$

The theorem is proved.

How to Obtain Variance from Membership Function?

Let ξ be an uncertain set with membership function μ. In order to calculate its variance by (8.231), we must determine the value of $\mathcal{M}\{(\xi-e)^2 \succeq x\}$ from the membership function μ.

Theorem 8.36 *(Liu [94]) Let ξ be a nonempty uncertain set with membership function μ. Then for any real numbers e and x, we have*

$$\mathcal{M}\{(\xi-e)^2 \succeq x\} = \frac{1}{2}\left(\sup_{(y-e)^2\ge x}\mu(y) + 1 - \sup_{(y-e)^2<x}\mu(y)\right). \tag{8.234}$$

Proof: Since ξ is an uncertain set with membership function μ, it follows from the measure inversion formula that for any real numbers e and x, we have

$$\mathcal{M}\{(\xi-e)^2 \ge x\} = 1 - \sup_{(y-e)^2<x}\mu(y),$$

$$\mathcal{M}\{(\xi-e)^2 < x\} = 1 - \sup_{(y-e)^2\ge x}\mu(y).$$

The equation (8.234) is thus proved by (8.232).

Theorem 8.37 *(Liu [94]) Let ξ be an uncertain set with membership function μ and finite expected value e. Then*

$$V[\xi] = \frac{1}{2}\int_0^{+\infty}\left(\sup_{(y-e)^2\ge x}\mu(y) + 1 - \sup_{(y-e)^2<x}\mu(y)\right)\mathrm{d}x. \tag{8.235}$$

Proof: This theorem follows from (8.231) and (8.234) immediately.

8.9 Distance

Definition 8.14 *(Liu [84]) The distance between nonempty uncertain sets ξ and η is defined as*

$$d(\xi,\eta) = E[|\xi-\eta|]. \tag{8.236}$$

That is, the distance between ξ and η is just the expected value of $|\xi-\eta|$. Since $|\xi-\eta|$ is a nonnegative uncertain set, we have

$$d(\xi,\eta) = \int_0^{+\infty}\mathcal{M}\{|\xi-\eta| \succeq x\}\mathrm{d}x. \tag{8.237}$$

Please note that $|\xi-\eta| \succeq x$ represents "$|\xi-\eta|$ is imaginarily included in $[x,+\infty)$". What is the appropriate value of $\mathcal{M}\{|\xi-\eta| \succeq x\}$? Intuitively, it is too conservative if we take the value $\mathcal{M}\{|\xi-\eta| \ge x\}$, and it is too adventurous if we take the value $1-\mathcal{M}\{|\xi-\eta| < x\}$. Thus we assign $\mathcal{M}\{|\xi-\eta| \succeq x\}$ the middle value between them. That is,

$$\mathcal{M}\{|\xi-\eta| \succeq x\} = \frac{1}{2}\left(\mathcal{M}\{|\xi-\eta| \ge x\} + 1 - \mathcal{M}\{|\xi-\eta| < x\}\right). \tag{8.238}$$

Theorem 8.38 *(Liu [94]) Let ξ and η be nonempty uncertain sets. Then for any real number x, we have*

$$\mathcal{M}\{|\xi-\eta| \succeq x\} = \frac{1}{2}\left(\sup_{|y|\ge x}\lambda(y) + 1 - \sup_{|y|<x}\lambda(y)\right) \tag{8.239}$$

where λ is the membership function of $\xi-\eta$.

Proof: Since $\xi-\eta$ is an uncertain set with membership function λ, it follows from the measure inversion formula that for any real number x, we have

$$\mathcal{M}\{|\xi-\eta| \ge x\} = 1 - \sup_{|y|<x}\mu(y),$$

$$\mathcal{M}\{|\xi-\eta| < x\} = 1 - \sup_{|y|\ge x}\mu(y).$$

The equation (8.239) is thus proved by (8.238).

Theorem 8.39 *(Liu [94]) Let ξ and η be nonempty uncertain sets. Then the distance between ξ and η is*

$$d(\xi,\eta) = \frac{1}{2}\int_0^{+\infty}\left(\sup_{|y|\ge x}\lambda(y) + 1 - \sup_{|y|<x}\lambda(y)\right)\mathrm{d}x \tag{8.240}$$

where λ is the membership function of $\xi-\eta$.

Proof: The theorem follows from (8.237) and (8.239) immediately.

8.10 Entropy

This section provides a definition of entropy to characterize the uncertainty of uncertain sets.

Definition 8.15 *(Liu [84]) Suppose that ξ is an uncertain set with membership function μ. Then its entropy is defined by*

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x \tag{8.241}$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

Remark 8.13: Note that the entropy (8.241) has the same form with de Luca and Termini's entropy for fuzzy set [23].

Remark 8.14: If ξ is a discrete uncertain set taking values in $\{x_1, x_2, \cdots\}$, then the entropy becomes

$$H[\xi] = \sum_{i=1}^{\infty} S(\mu(x_i)). \tag{8.242}$$

Example 8.35: A crisp set A of real numbers is a special uncertain set $\xi(\gamma) \equiv A$. Its membership function is

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

and entropy is

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x = \int_{-\infty}^{+\infty} 0\mathrm{d}x = 0.$$

This means a crisp set has no uncertainty.

Exercise 8.54: Let $\xi = (a, b, c)$ be a triangular uncertain set. Show that its entropy is

$$H[\xi] = \frac{c-a}{2}. \tag{8.243}$$

Exercise 8.55: Let $\xi = (a, b, c, d)$ be a trapezoidal uncertain set. Show that its entropy is

$$H[\xi] = \frac{b-a+d-c}{2}. \tag{8.244}$$

Theorem 8.40 *Let ξ be an uncertain set. Then $H[\xi] \geq 0$ and equality holds if ξ is essentially a crisp set.*

Proof: The nonnegativity is clear. In addition, when an uncertain set tends to a crisp set, its entropy tends to the minimum value 0.

Theorem 8.41 *Let ξ be an uncertain set on the interval $[a, b]$. Then*

$$H[\xi] \leq (b-a)\ln 2 \tag{8.245}$$

and equality holds if ξ has a membership function $\mu(x) = 0.5$ on $[a, b]$.

Proof: The theorem follows from the fact that the function $S(t)$ reaches its maximum value $\ln 2$ at $t = 0.5$.

Theorem 8.42 *Let ξ be an uncertain set, and let ξ^c be its complement. Then*

$$H[\xi^c] = H[\xi]. \tag{8.246}$$

Proof: Write the membership function of ξ by μ. Then its complement ξ^c has a membership function $1 - \mu(x)$. It follows from the definition of entropy that

$$H[\xi^c] = \int_{-\infty}^{+\infty} S\left(1 - \mu(x)\right)\mathrm{d}x = \int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x = H[\xi].$$

The theorem is proved.

Theorem 8.43 *(Yao-Ke [172]) Let ξ be an uncertain set with regular membership function μ. Then*

$$H[\xi] = \int_0^1 (\mu_l^{-1}(\alpha) - \mu_r^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha. \tag{8.247}$$

Proof: It is clear that $S(\alpha) = -\alpha \ln \alpha - (1-\alpha)\ln(1-\alpha)$ is a derivable function whose derivative is

$$S'(\alpha) = -\ln \frac{\alpha}{1-\alpha}.$$

Let x_0 be a point such that $\mu(x_0) = 1$. Then we have

$$\begin{aligned} H[\xi] &= \int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x = \int_{-\infty}^{x_0} S(\mu(x))\mathrm{d}x + \int_{x_0}^{+\infty} S(\mu(x))\mathrm{d}x \\ &= \int_{-\infty}^{x_0} \int_0^{\mu(x)} S'(\alpha)\mathrm{d}\alpha\mathrm{d}x + \int_{x_0}^{+\infty} \int_0^{\mu(x)} S'(\alpha)\mathrm{d}\alpha\mathrm{d}x. \end{aligned}$$

It follows from Fubini theorem that

$$\begin{aligned} H[\xi] &= \int_0^1 \int_{\mu_l^{-1}(\alpha)}^{x_0} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha + \int_0^1 \int_{x_0}^{\mu_r^{-1}(\alpha)} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha \\ &= \int_0^1 (x_0 - \mu_l^{-1}(\alpha))S'(\alpha)\mathrm{d}\alpha + \int_0^1 (\mu_r^{-1}(\alpha) - x_0)S'(\alpha)\mathrm{d}\alpha \\ &= \int_0^1 (\mu_r^{-1}(\alpha) - \mu_l^{-1}(\alpha))S'(\alpha)\mathrm{d}\alpha \\ &= \int_0^1 (\mu_l^{-1}(\alpha) - \mu_r^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha}\mathrm{d}\alpha. \end{aligned}$$

The theorem is verified.

Positive Linearity of Entropy

Theorem 8.44 *(Yao-Ke [172]) Let ξ and η be independent uncertain sets. Then for any real numbers a and b, we have*

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta]. \tag{8.248}$$

Proof: Without loss of generality, assume the uncertain sets ξ and η have regular membership functions μ and ν, respectively.

STEP 1: We prove $H[a\xi] = |a|H[\xi]$. If $a > 0$, then the left and right inverse membership functions of $a\xi$ are

$$\lambda_l^{-1}(\alpha) = a\mu_l^{-1}(\alpha), \quad \lambda_r^{-1}(\alpha) = a\mu_r^{-1}(\alpha).$$

It follows from Theorem 8.43 that

$$H[a\xi] = \int_0^1 (a\mu_l^{-1}(\alpha) - a\mu_r^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha = aH[\xi] = |a|H[\xi].$$

If $a = 0$, then we immediately have $H[a\xi] = 0 = |a|H[\xi]$. If $a < 0$, then we have

$$\lambda_l^{-1}(\alpha) = a\mu_r^{-1}(\alpha), \quad \lambda_r^{-1}(\alpha) = a\mu_l^{-1}(\alpha)$$

and

$$H[a\xi] = \int_0^1 (a\mu_r^{-1}(\alpha) - a\mu_l^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha = (-a)H[\xi] = |a|H[\xi].$$

Thus we always have $H[a\xi] = |a|H[\xi]$.

STEP 2: We prove $H[\xi + \eta] = H[\xi] + H[\eta]$. Note that the left and right inverse membership functions of $\xi + \eta$ are

$$\lambda_l^{-1}(\alpha) = \mu_l^{-1}(\alpha) + \nu_l^{-1}(\alpha), \quad \lambda_r^{-1}(\alpha) = \mu_r^{-1}(\alpha) + \nu_r^{-1}(\alpha).$$

It follows from Theorem 8.43 that

$$\begin{aligned} H[\xi + \eta] &= \int_0^1 (\lambda_l^{-1}(\alpha) - \lambda_r^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha \\ &= \int_0^1 (\mu_l^{-1}(\alpha) + \nu_l^{-1}(\alpha) - \mu_r^{-1}(\alpha) - \nu_r^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha \\ &= H[\xi] + H[\eta]. \end{aligned}$$

STEP 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$H[a\xi + b\eta] = H[a\xi] + H[b\eta] = |a|H[\xi] + |b|H[\eta].$$

The theorem is proved.

Exercise 8.56: Let ξ be an uncertain set, and let A be a crisp set. Show that

$$H[\xi + A] = H[\xi]. \tag{8.249}$$

That is, the entropy is invariant under arbitrary translations.

Example 8.36: The independence condition in Theorem 8.44 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then

$$\xi(\gamma) = [-\gamma, \gamma] \tag{8.250}$$

is a triangular uncertain set $(-1, 0, 1)$ with entropy

$$H[\xi] = 1, \tag{8.251}$$

and

$$\eta(\gamma) = [\gamma - 1, 1 - \gamma] \tag{8.252}$$

is also a triangular uncertain set $(-1, 0, 1)$ with entropy

$$H[\eta] = 1. \tag{8.253}$$

Note that ξ and η are not independent, and $\xi + \eta \equiv [-1, 1]$ whose entropy is

$$H[\xi + \eta] = 0. \tag{8.254}$$

Thus

$$H[\xi + \eta] \neq H[\xi] + H[\eta]. \tag{8.255}$$

Therefore, the independence condition cannot be removed.

8.11 Conditional Membership Function

What is the conditional membership function of an uncertain set ξ after it has been learned that some event A has occurred? This section will answer this question. At first, it follows from the definition of conditional uncertain measure that

$$\mathcal{M}\{B \subset \xi | A\} = \begin{cases} \dfrac{\mathcal{M}\{(B \subset \xi) \cap A\}}{\mathcal{M}\{A\}}, & \text{if } \dfrac{\mathcal{M}\{(B \subset \xi) \cap A\}}{\mathcal{M}\{A\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{(B \not\subset \xi) \cap A\}}{\mathcal{M}\{A\}}, & \text{if } \dfrac{\mathcal{M}\{(B \not\subset \xi) \cap A\}}{\mathcal{M}\{A\}} < 0.5 \\ 0.5, & \text{otherwise,} \end{cases}$$

$$\mathcal{M}\{\xi \subset B | A\} = \begin{cases} \dfrac{\mathcal{M}\{(\xi \subset B) \cap A\}}{\mathcal{M}\{A\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi \subset B) \cap A\}}{\mathcal{M}\{A\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{(\xi \not\subset B) \cap A\}}{\mathcal{M}\{A\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi \not\subset B) \cap A\}}{\mathcal{M}\{A\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Definition 8.16 *(Liu [94]) Let ξ be an uncertain set, and let A be an event with $\mathcal{M}\{A\} > 0$. Then the conditional uncertain set ξ given A is said to have a membership function $\mu(x|A)$ if for any Borel set B of real numbers, we have*

$$\mathcal{M}\{B \subset \xi | A\} = \inf_{x \in B} \mu(x|A), \tag{8.256}$$

$$\mathcal{M}\{\xi \subset B | A\} = 1 - \sup_{x \in B^c} \mu(x|A). \tag{8.257}$$

Theorem 8.45 *(Yao [183]) Let ξ be a totally ordered uncertain set on a continuous uncertainty space, and let A be an event with $\mathcal{M}\{A\} > 0$. Then the conditional membership function of ξ given A exists, and*

$$\mu(x|A) = \mathcal{M}\{x \in \xi|A\}. \tag{8.258}$$

Proof: Since the original uncertain measure $\mathcal{M}$ is continuous, the conditional uncertain measure $\mathcal{M}\{\cdot|A\}$ is also continuous. Thus the conditional uncertain set ξ given A is a totally ordered uncertain set on a continuous uncertainty space. It follows from Theorem 8.14 that the conditional membership function exists, and $\mu(x|A) = \mathcal{M}\{x \in \xi|A\}$. The proof is complete.

Example 8.37: The total order condition in Theorem 8.45 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise.} \end{cases} \tag{8.259}$$

Then

$$\xi(\gamma) = \begin{cases} [1, 4], & \text{if } \gamma = \gamma_1 \\ [1, 3], & \text{if } \gamma = \gamma_2 \\ [2, 4], & \text{if } \gamma = \gamma_3 \\ [2, 3], & \text{if } \gamma = \gamma_4 \end{cases} \tag{8.260}$$

is a non-totally ordered uncertain set on a continuous uncertainty space, but has a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } 2 \le x \le 3 \\ 0.5, & \text{if } 1 \le x < 2 \text{ or } 3 < x \le 4 \\ 0, & \text{otherwise.} \end{cases} \tag{8.261}$$

However, the conditional uncertain measure given $A = \{\gamma_1, \gamma_2, \gamma_3\}$ is

$$\mathcal{M}\{\Lambda|A\} = \begin{cases} 0, & \text{if } \Lambda \cap A = \emptyset \\ 1, & \text{if } \Lambda \supset A \\ 0.5, & \text{otherwise.} \end{cases} \tag{8.262}$$

If the conditional uncertain set ξ given A has a membership function, then

$$\mu(x|A) = \begin{cases} 1, & \text{if } 2 \le x \le 3 \\ 0.5, & \text{if } 1 \le x < 2 \text{ or } 3 < x \le 4 \\ 0, & \text{otherwise.} \end{cases} \tag{8.263}$$

Taking $B = [1.5, 3.5]$, we obtain

$$\mathcal{M}\{\xi \subset B|A\} = \mathcal{M}\{\gamma_4|A\} = 0 \neq 0.5 = 1 - \sup_{x \in B^c} \mu(x|A). \tag{8.264}$$

That is, the second measure inversion formula is not valid and then the conditional membership function does not exist. Thus the total order condition cannot be removed.

Example 8.38: The continuity condition in Theorem 8.45 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise.} \end{cases} \tag{8.265}$$

Then

$$\xi(\gamma) = (-\gamma, \gamma), \quad \forall \gamma \in [0, 1] \tag{8.266}$$

is a totally ordered uncertain set on a discontinuous uncertainty space, but has a membership function

$$\mu(x) = \begin{cases} 0.5, & \text{if } -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{8.267}$$

However, the conditional uncertain measure given $A = (0, 1)$ is

$$\mathcal{M}\{\Lambda|A\} = \begin{cases} 0, & \text{if } \Lambda \cap A = \emptyset \\ 1, & \text{if } \Lambda \supset A \\ 0.5, & \text{otherwise.} \end{cases} \tag{8.268}$$

If the conditional uncertain set ξ given A has a membership function, then

$$\mu(x|A) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } -1 < x < 0 \text{ or } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{8.269}$$

Taking $B = (-1, 1)$, we obtain

$$\mathcal{M}\{B \subset \xi|A\} = \mathcal{M}\{1|A\} = 0 \neq 0.5 = \inf_{x \in B} \mu(x|A). \tag{8.270}$$

That is, the first measure inversion formula is not valid and then the conditional membership function does not exist. Thus the continuity condition cannot be removed.

Theorem 8.46 *(Yao [183]) Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then for any real number a, the conditional uncertain set η given $a \in \xi$ has a membership function*

$$\nu(y|a \in \xi) = \begin{cases} \dfrac{\nu(y)}{\mu(a)}, & \textit{if } \nu(y) < \mu(a)/2 \\ \dfrac{\nu(y) + \mu(a) - 1}{\mu(a)}, & \textit{if } \nu(y) > 1 - \mu(a)/2 \\ 0.5, & \textit{otherwise.} \end{cases} \tag{8.271}$$

Proof: In order prove that $\nu(y|a \in \xi)$ is the membership function of the conditional uncertain set η given $a \in \xi$, we must verify the two measure inversion formulas,

$$\mathcal{M}\{B \subset \eta | a \in \xi\} = \inf_{y \in B} \nu(y|a \in \xi), \tag{8.272}$$

$$\mathcal{M}\{\eta \subset B | a \in \xi\} = 1 - \sup_{y \in B^c} \nu(y|a \in \xi). \tag{8.273}$$

First, for any Borel set B of real numbers, by using the definition of conditional uncertainty and independence of ξ and η, we have

$$\mathcal{M}\{B \subset \eta | a \in \xi\} = \begin{cases} \dfrac{\mathcal{M}\{B \subset \eta\}}{\mathcal{M}\{a \in \xi\}}, & \text{if } \dfrac{\mathcal{M}\{B \subset \eta\}}{\mathcal{M}\{a \in \xi\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{B \not\subset \eta\}}{\mathcal{M}\{a \in \xi\}}, & \text{if } \dfrac{\mathcal{M}\{B \not\subset \eta\}}{\mathcal{M}\{a \in \xi\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Since

$$\mathcal{M}\{B \subset \eta\} = \inf_{y \in B} \nu(y), \quad \mathcal{M}\{B \not\subset \eta\} = 1 - \inf_{y \in B} \nu(y), \quad \mathcal{M}\{a \in \xi\} = \mu(a),$$

we get

$$\mathcal{M}\{B \subset \eta | a \in \xi\} = \begin{cases} \dfrac{\inf\limits_{y \in B} \nu(y)}{\mu(a)}, & \text{if } \inf\limits_{y \in B} \nu(y) < \mu(a)/2 \\ \dfrac{\inf\limits_{y \in B} \nu(y) + \mu(a) - 1}{\mu(a)}, & \text{if } \inf\limits_{y \in B} \nu(y) > 1 - \mu(a)/2 \\ 0.5, & \text{otherwise.} \end{cases}$$

That is,

$$\mathcal{M}\{B \subset \eta | a \in \xi\} = \inf_{y \in B} \nu(y|a \in \xi).$$

The first measure inversion formula is verified. Next, by using the definition of conditional uncertainty and independence of ξ and η, we have

$$\mathcal{M}\{\eta \subset B | a \in \xi\} = \begin{cases} \dfrac{\mathcal{M}\{\eta \subset B\}}{\mathcal{M}\{a \in \xi\}}, & \text{if } \dfrac{\mathcal{M}\{\eta \subset B\}}{\mathcal{M}\{a \in \xi\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{\eta \not\subset B\}}{\mathcal{M}\{a \in \xi\}}, & \text{if } \dfrac{\mathcal{M}\{\eta \not\subset B\}}{\mathcal{M}\{a \in \xi\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Since

$$\mathcal{M}\{\eta \subset B\} = 1 - \sup_{y \in B^c} \nu(y), \quad \mathcal{M}\{\eta \not\subset B\} = \sup_{y \in B^c} \nu(y), \quad \mathcal{M}\{a \in \xi\} = \mu(a),$$

we get

$$\mathcal{M}\{\eta \subset B | a \in \xi\} = \begin{cases} \dfrac{1 - \sup_{y \in B^c} \nu(y)}{\mu(a)}, & \text{if } \sup_{y \in B^c} \nu(y) > 1 - \mu(a)/2 \\ \dfrac{\mu(a) - \sup_{y \in B^c} \nu(y)}{\mu(a)}, & \text{if } \sup_{y \in B^c} \nu(y) < \mu(a)/2 \\ 0.5, & \text{otherwise.} \end{cases}$$

That is,

$$\mathcal{M}\{\eta \subset B | a \in \xi\} = 1 - \sup_{y \in B^c} \nu(y | a \in \xi).$$

The second measure inversion formula is verified. Hence $\nu(y|a \in \xi)$ is the membership function of the conditional uncertain set η given $a \in \xi$.

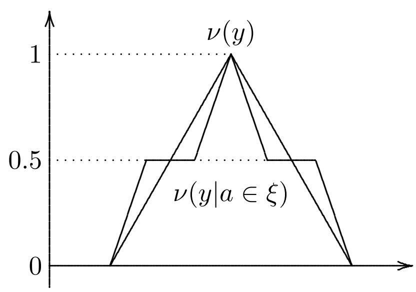


Figure 8.14: Membership Functions $\nu(y)$ and $\nu(y|a \in \xi)$

Exercise 8.57: Let $\xi_1, \xi_2, \cdots, \xi_m, \eta$ be independent uncertain sets with membership functions $\mu_1, \mu_2, \cdots, \mu_m, \nu$, respectively. For any real numbers

$a_1, a_2, \cdots, a_m$, show that the conditional uncertain set η given $a_1 \in \xi_1, a_2 \in \xi_2, \cdots, a_m \in \xi_m$ has a membership function

$$\nu^*(y) = \begin{cases} \dfrac{\nu(y)}{\min\limits_{1\le i\le m} \mu_i(a_i)}, & \text{if } \nu(y) < \min\limits_{1\le i\le m} \mu_i(a_i)/2 \\ \dfrac{\nu(y) + \min\limits_{1\le i\le m} \mu_i(a_i) - 1}{\min\limits_{1\le i\le m} \mu_i(a_i)}, & \text{if } \nu(y) > 1 - \min\limits_{1\le i\le m} \mu_i(a_i)/2 \\ 0.5, & \text{otherwise.} \end{cases}$$

8.12 Bibliographic Notes

In order to model unsharp concepts like "young", "tall" and "most", uncertain set was proposed by Liu [81] in 2010. After that, membership function was presented by Liu [87] in 2012 to describe uncertain sets. However, not all uncertain sets have membership functions. Liu [98] proved that totally ordered uncertain sets on a continuous uncertainty space always have membership functions. In addition, Liu [90] defined the independence of uncertain sets, and provided the operational law through membership functions. Yao [177] derived a formula for calculating the uncertain measure of inclusion relation between uncertain sets.

The expected value of uncertain set was defined by Liu [81]. Next, Liu [83] gave a formula for caluculating the expected value by membership function, and Liu [87] provided a formula by inverse membership function. Based on the expected value operator, Liu [84] presented the variance and distance between uncertain sets, and Yang-Gao [156] investigated the moments of uncertain set.

The entropy was presented by Liu [84] for measuring the uncertainty of uncertain set. Some formulas were also provided by Yao-Ke [172] for calculating the value of entropy.

Conditional uncertain set was first investigated by Liu [81] and conditional membership function was formally defined by Liu [94]. Furthermore, Yao [183] presented some criteria for judging the existence of conditional membership function.

Chapter 9

Uncertain Logic

Uncertain logic is a methodology for calculating the truth values of uncertain propositions via uncertain set theory. This chapter will introduce individual feature data, uncertain quantifier, uncertain subject, uncertain predicate, uncertain proposition, and truth value. Uncertain logic may provide a flexible means for extracting linguistic summary from a collection of raw data.

9.1 Individual Feature Data

At first, we should have a universe A of individuals we are talking about. Without loss of generality, we may assume that A consists of n individuals and is represented by

$$A = \{a_1, a_2, \cdots, a_n\}. \tag{9.1}$$

In order to deal with the universe A, we should have feature data of all individuals $a_1, a_2, \cdots, a_n$. When we talk about "those days are warm", we should know the individual feature data of all days, for example,

$$A = \{22,\ 23,\ 25,\ 28,\ 30,\ 32,\ 36\} \tag{9.2}$$

whose elements are temperatures in centigrades. When we talk about "those students are young", we should know the individual feature data of all students, for example,

$$A = \{21,\ 22,\ 22,\ 23,\ 24,\ 25,\ 26,\ 27,\ 28,\ 30,\ 32,\ 35,\ 36,\ 38,\ 40\} \tag{9.3}$$

whose elements are ages in years. When we talk about "those sportsmen are tall", we should know the individual feature data of all sportsmen, for example,

$$A = \left\{\begin{array}{c} 175,\ 178,\ 178,\ 180,\ 183,\ 184,\ 186,\ 186 \\ 188,\ 190,\ 192,\ 192,\ 193,\ 194,\ 195,\ 196 \end{array}\right\} \tag{9.4}$$

whose elements are heights in centimeters.

Sometimes the individual feature data are represented by vectors rather a scalar number. When we talk about "those young students are tall", we should know the individual feature data of all students, for example,

$$A = \left\{ \begin{array}{l} (24,185),\ (25,190),\ (26,184),\ (26,170),\ (27,187),\ (27,188) \\ (28,160),\ (30,190),\ (32,185),\ (33,176),\ (35,185),\ (36,188) \\ (38,164),\ (38,178),\ (39,182),\ (40,186),\ (42,165),\ (44,170) \end{array} \right\} \tag{9.5}$$

whose elements are ages and heights in years and centimeters, respectively.

9.2 Uncertain Quantifier

If we want to represent all individuals in the universe A, we use the universal quantifier ($\forall$) and

$$\forall = \text{"for all"}. \tag{9.6}$$

If we want to represent some (at least one) individuals, we use the existential quantifier ($\exists$) and

$$\exists = \text{"there exists at least one"}. \tag{9.7}$$

In addition to the two quantifiers, there are numerous imprecise quantifiers in human language, for example, *almost all*, *almost none*, *many*, *several*, *some*, *most*, *a few*, *about half*. This section will model them by the tool of uncertain quantifier.

Definition 9.1 *(Liu [84]) Uncertain quantifier is an uncertain set representing the number of individuals.*

Example 9.1: The universal quantifier ($\forall$) on the universe A is a special uncertain quantifier,

$$\forall \equiv \{n\} \tag{9.8}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } x = n \\ 0, & \text{otherwise.} \end{cases} \tag{9.9}$$

Example 9.2: The existential quantifier ($\exists$) on the universe A is a special uncertain quantifier,

$$\exists \equiv \{1, 2, \cdots, n\} \tag{9.10}$$

whose membership function is

$$\lambda(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{otherwise.} \end{cases} \tag{9.11}$$

Example 9.3: The quantifier "*there does not exist one*" on the universe A is a special uncertain quantifier

$$\mathcal{Q} \equiv \{0\} \tag{9.12}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{9.13}$$

Example 9.4: The quantifier "*there exist exactly* m" on the universe A is a special uncertain quantifier

$$\mathcal{Q} \equiv \{m\} \tag{9.14}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } x = m \\ 0, & \text{otherwise.} \end{cases} \tag{9.15}$$

Example 9.5: The quantifier "*there exist at least* m" on the universe A is a special uncertain quantifier

$$\mathcal{Q} \equiv \{m, m+1, \cdots, n\} \tag{9.16}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } m \le x \le n \\ 0, & \text{if } 0 \le x < m. \end{cases} \tag{9.17}$$

Example 9.6: The quantifier "*there exist at most* m" on the universe A is a special uncertain quantifier

$$\mathcal{Q} \equiv \{0, 1, 2, \cdots, m\} \tag{9.18}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le m \\ 0, & \text{if } m < x \le n. \end{cases} \tag{9.19}$$

Example 9.7: The uncertain quantifier $\mathcal{Q}$ of "*almost all*" on the universe A may have a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le n-5 \\ (x-n+5)/3, & \text{if } n-5 \le x \le n-2 \\ 1, & \text{if } n-2 \le x \le n. \end{cases} \tag{9.20}$$

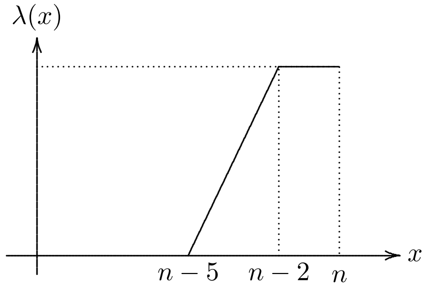


Figure 9.1: Membership Function of Quantifier "*almost all*"

Example 9.8: The uncertain quantifier $\mathcal{Q}$ of "*almost none*" on the universe A may have a membership function

$$\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le 2 \\ (5-x)/3, & \text{if } 2 \le x \le 5 \\ 0, & \text{if } 5 \le x \le n. \end{cases} \tag{9.21}$$

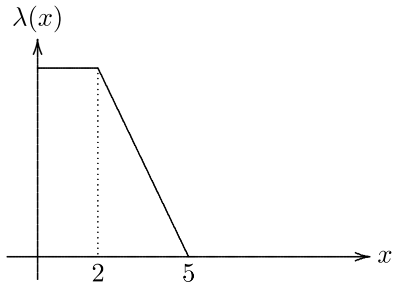


Figure 9.2: Membership Function of Quantifier "*almost none*"

Example 9.9: The uncertain quantifier $\mathcal{Q}$ of "*about 10*" on the universe A may have a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 7 \\ (x-7)/2, & \text{if } 7 \le x \le 9 \\ 1, & \text{if } 9 \le x \le 11 \\ (13-x)/2, & \text{if } 11 \le x \le 13 \\ 0, & \text{if } 13 \le x \le n. \end{cases} \tag{9.22}$$

Example 9.10: In many cases, it is more convenient for us to use a percentage than an absolute quantity. For example, we may use the uncertain

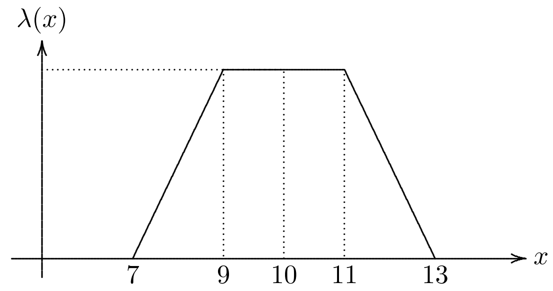


Figure 9.3: Membership Function of Quantifier "*about 10*"

quantifier $\mathcal{Q}$ of "*about 70%*". In this case, a possible membership function of $\mathcal{Q}$ is

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 0.6 \\ 20(x-0.6), & \text{if } 0.6 \leq x \leq 0.65 \\ 1, & \text{if } 0.65 \leq x \leq 0.75 \\ 20(0.8-x), & \text{if } 0.75 \leq x \leq 0.8 \\ 0, & \text{if } 0.8 \leq x \leq 1. \end{cases} \tag{9.23}$$

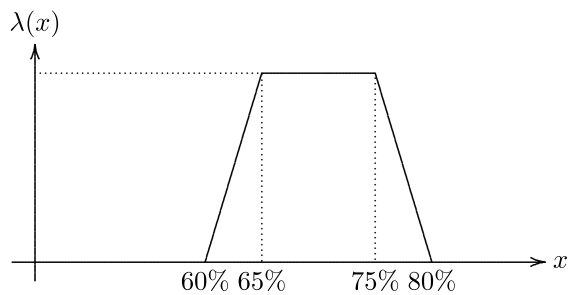


Figure 9.4: Membership Function of Quantifier "*about 70%*"

Definition 9.2 *An uncertain quantifier is said to be unimodal if its membership function is unimodal.*

Example 9.11: The uncertain quantifiers "almost all", "almost none", "about 10" and "about 70%" are unimodal.

Definition 9.3 *An uncertain quantifier is said to be monotone if its membership function is monotone. Especially, an uncertain quantifier is said to be increasing if its membership function is increasing; and an uncertain quantifier is said to be decreasing if its membership function is decreasing.*

The uncertain quantifiers "almost all" and "almost none" are monotone, but "about 10" and "about 70%" are not monotone. Note that both increasing uncertain quantifiers and decreasing uncertain quantifiers are monotone. In addition, any monotone uncertain quantifiers are unimodal.

Negated Quantifier

What is the negation of an uncertain quantifier? The following definition gives a formal answer.

Definition 9.4 *(Liu [84]) Let $\mathcal{Q}$ be an uncertain quantifier. Then the negated quantifier $\neg\mathcal{Q}$ is the complement of $\mathcal{Q}$ in the sense of uncertain set, i.e.,*

$$\neg\mathcal{Q} = \mathcal{Q}^c. \tag{9.24}$$

Example 9.12: Let $\forall = \{n\}$ be the universal quantifier. Then its negated quantifier

$$\neg\forall \equiv \{0, 1, 2, \cdots, n-1\}. \tag{9.25}$$

Example 9.13: Let $\exists = \{1, 2, \cdots, n\}$ be the existential quantifier. Then its negated quantifier is

$$\neg\exists \equiv \{0\}. \tag{9.26}$$

Theorem 9.1 *Let $\mathcal{Q}$ be an uncertain quantifier whose membership function is λ. Then the negated quantifier $\neg\mathcal{Q}$ has a membership function*

$$\neg\lambda(x) = 1 - \lambda(x). \tag{9.27}$$

Proof: This theorem follows from the operational law of uncertain set immediately.

Example 9.14: Let $\mathcal{Q}$ be the uncertain quantifier "*almost all*" defined by (9.20). Then its negated quantifier $\neg\mathcal{Q}$ has a membership function

$$\neg\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le n-5 \\ (n-x-2)/3, & \text{if } n-5 \le x \le n-2 \\ 0, & \text{if } n-2 \le x \le n. \end{cases} \tag{9.28}$$

Example 9.15: Let $\mathcal{Q}$ be the uncertain quantifier "*about 70%*" defined by (9.23). Then its negated quantifier $\neg\mathcal{Q}$ has a membership function

$$\neg\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le 0.6 \\ 20(0.65-x), & \text{if } 0.6 \le x \le 0.65 \\ 0, & \text{if } 0.65 \le x \le 0.75 \\ 20(x-0.75), & \text{if } 0.75 \le x \le 0.8 \\ 1, & \text{if } 0.8 \le x \le 1. \end{cases} \tag{9.29}$$

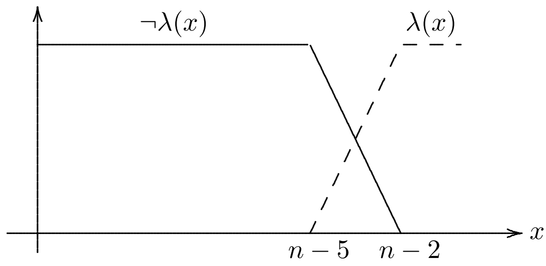


Figure 9.5: Membership Function of Negated Quantifier of "*almost all*"

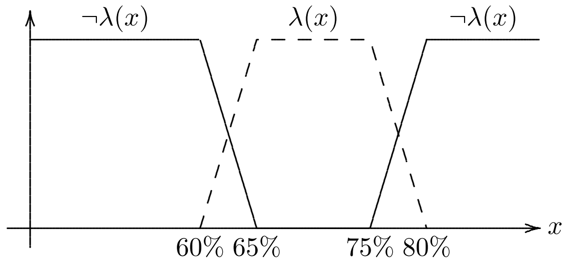


Figure 9.6: Membership Function of Negated Quantifier of "*about 70%*"

Theorem 9.2 *Let $\mathcal{Q}$ be an uncertain quantifier. Then we have $\neg\neg\mathcal{Q} = \mathcal{Q}$.*

Proof: This theorem follows from $\neg\neg\mathcal{Q} = \neg\mathcal{Q}^c = (\mathcal{Q}^c)^c = \mathcal{Q}$.

Theorem 9.3 *If $\mathcal{Q}$ is a monotone uncertain quantifier, then $\neg\mathcal{Q}$ is also monotone. Especially, if $\mathcal{Q}$ is increasing, then $\neg\mathcal{Q}$ is decreasing; if $\mathcal{Q}$ is decreasing, then $\neg\mathcal{Q}$ is increasing.*

Proof: Assume λ is the membership function of $\mathcal{Q}$. Then $\neg\mathcal{Q}$ has a membership function $1 - \lambda(x)$. The theorem follows from this fact immediately.

Dual Quantifier

Definition 9.5 *(Liu [84]) Let $\mathcal{Q}$ be an uncertain quantifier. Then the dual quantifier of $\mathcal{Q}$ is*

$$\mathcal{Q}^* = \forall - \mathcal{Q}. \tag{9.30}$$

Remark 9.1: Note that $\mathcal{Q}$ and $\mathcal{Q}^*$ are dependent uncertain sets such that $\mathcal{Q} + \mathcal{Q}^* \equiv \forall$. Since the cardinality of the universe A is n, we also have

$$\mathcal{Q}^* = \{n\} - \mathcal{Q}. \tag{9.31}$$

Example 9.16: Since $\forall \equiv \{n\}$, we immediately have $\forall^* = \{0\} = \neg\exists$. That is

$$\forall^* \equiv \neg\exists. \tag{9.32}$$

Example 9.17: Since $\neg\forall = \{0, 1, 2, \cdots, n-1\}$, we immediately have $(\neg\forall)^* = \{1, 2, \cdots, n\} = \exists$. That is,

$$(\neg\forall)^* \equiv \exists. \tag{9.33}$$

Example 9.18: Since $\exists \equiv \{1, 2, \cdots, n\}$, we have $\exists^* = \{0, 1, 2, \cdots, n-1\} = \neg\forall$. That is,

$$\exists^* \equiv \neg\forall. \tag{9.34}$$

Example 9.19: Since $\neg\exists = \{0\}$, we immediately have $(\neg\exists)^* = \{n\} = \forall$. That is,

$$(\neg\exists)^* = \forall. \tag{9.35}$$

Theorem 9.4 *Let $\mathcal{Q}$ be an uncertain quantifier whose membership function is λ. Then the dual quantifier $\mathcal{Q}^*$ has a membership function*

$$\lambda^*(x) = \lambda(n - x) \tag{9.36}$$

where n is the cardinality of the universe A.

Proof: This theorem follows from the operational law of uncertain set immediately.

Example 9.20: Let $\mathcal{Q}$ be the uncertain quantifier "*almost all*" defined by (9.20). Then its dual quantifier $\mathcal{Q}^*$ has a membership function

$$\lambda^*(x) = \begin{cases} 1, & \text{if } 0 \le x \le 2 \\ (5-x)/3, & \text{if } 2 \le x \le 5 \\ 0, & \text{if } 5 \le x \le n. \end{cases} \tag{9.37}$$

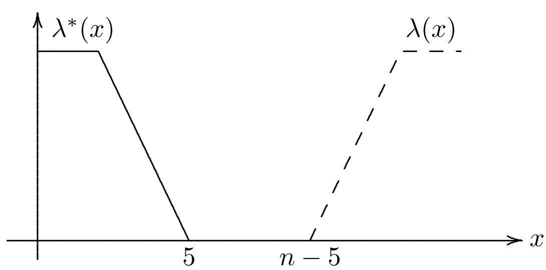


Figure 9.7: Membership Function of Dual Quantifier of "*almost all*"

Example 9.21: Let $\mathbb{Q}$ be the uncertain quantifier "*about 70%*" defined by (9.23). Then its dual quantifier $\mathbb{Q}^*$ has a membership function

$$\lambda^*(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.2 \\ 20(x-0.2), & \text{if } 0.2 \le x \le 0.25 \\ 1, & \text{if } 0.25 \le x \le 0.35 \\ 20(0.4-x), & \text{if } 0.35 \le x \le 0.4 \\ 0, & \text{if } 0.4 \le x \le 1. \end{cases} \tag{9.38}$$

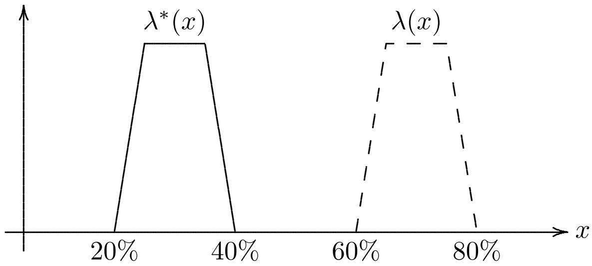


Figure 9.8: Membership Function of Dual Quantifier of "*about 70%*"

Theorem 9.5 *Let $\mathbb{Q}$ be an uncertain quantifier. Then we have $\mathbb{Q}^{**} = \mathbb{Q}$.*

Proof: The theorem follows from $\mathbb{Q}^{**} = \forall - \mathbb{Q}^* = \forall - (\forall - \mathbb{Q}) = \mathbb{Q}$.

Theorem 9.6 *If $\mathbb{Q}$ is a unimodal uncertain quantifier, then $\mathbb{Q}^*$ is also unimodal. Especially, if $\mathbb{Q}$ is a monotone, then $\mathbb{Q}^*$ is monotone; if $\mathbb{Q}$ is increasing, then $\mathbb{Q}^*$ is decreasing; if $\mathbb{Q}$ is decreasing, then $\mathbb{Q}^*$ is increasing.*

Proof: Assume λ is the membership function of $\mathbb{Q}$. Then $\mathbb{Q}^*$ has a membership function $\lambda(n-x)$. The theorem follows from this fact immediately.

9.3 Uncertain Subject

Sometimes, we are interested in a subset of the universe of individuals, for example, "warm days", "young students" and "tall sportsmen". This section will model them by the concept of uncertain subject.

Definition 9.6 *(Liu [84]) Uncertain subject is an uncertain set containing some specified individuals in the universe.*

Example 9.22: "Warm days are here again" is a statement in which "warm days" is an uncertain subject that is an uncertain set on the universe of "all days", whose membership function may be defined by

$$\nu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/3, & \text{if } 15 \le x \le 18 \\ 1, & \text{if } 18 \le x \le 24 \\ (28-x)/4, & \text{if } 24 \le x \le 28 \\ 0, & \text{if } 28 \le x. \end{cases} \tag{9.39}$$

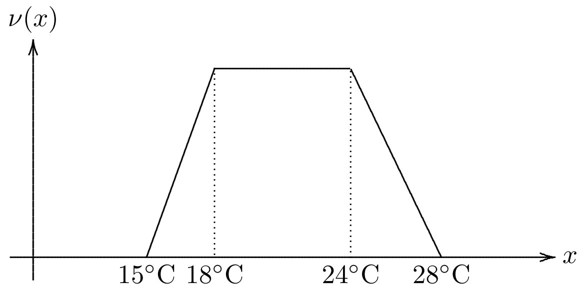


Figure 9.9: Membership Function of Subject "warm days"

Example 9.23: "Young students are tall" is a statement in which "young students" is an uncertain subject that is an uncertain set on the universe of "all students", whose membership function may be defined by

$$\nu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 35 \\ (45-x)/10, & \text{if } 35 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{9.40}$$

Example 9.24: "Tall students are heavy" is a statement in which "tall students" is an uncertain subject that is an uncertain set on the universe of "all students", whose membership function may be defined by

$$\nu(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x-180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } 185 \le x \le 195 \\ (200-x)/5, & \text{if } 195 \le x \le 200 \\ 0, & \text{if } x \ge 200. \end{cases} \tag{9.41}$$

Let S be an uncertain subject with membership function ν on the universe $A = \{a_1, a_2, \cdots, a_n\}$ of individuals. Then S is an uncertain set of A such

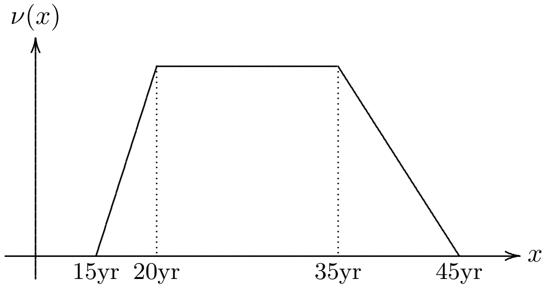


Figure 9.10: Membership Function of Subject "young students"

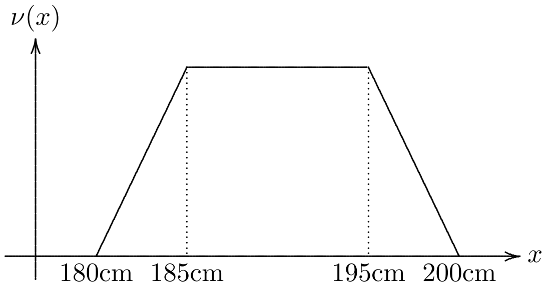


Figure 9.11: Membership Function of Subject "tall students"

that

$$\mathcal{M}\{a_i \in S\} = \nu(a_i), \quad i = 1, 2, \cdots, n. \tag{9.42}$$

In many cases, we are interested in some individuals a's with $\nu(a) \geq \omega$, where ω is a confidence level. Thus we have a subuniverse,

$$S_\omega = \{a \in A \mid \nu(a) \geq \omega\} \tag{9.43}$$

that will play a new universe of individuals we are talking about, and the individuals out of S_ω will be ignored at the confidence level ω.

Theorem 9.7 *Let ω_1 and ω_2 be confidence levels with $\omega_1 > \omega_2$, and let S_{ω_1} and S_{ω_2} be subuniverses with confidence levels ω_1 an ω_2, respectively. Then*

$$S_{\omega_1} \subset S_{\omega_2}. \tag{9.44}$$

That is, S_ω is a decreasing sequence of sets with respect to ω.

Proof: If $a \in S_{\omega_1}$, then $\nu(a) \geq \omega_1 > \omega_2$. Thus $a \in S_{\omega_2}$. It follows that $S_{\omega_1} \subset S_{\omega_2}$. Note that S_{ω_1} and S_{ω_2} may be empty.

9.4 Uncertain Predicate

There are numerous imprecise predicates in human language, for example, *warm*, *cold*, *hot*, *young*, *old*, *tall*, *small*, and *big*. This section will model them by the concept of uncertain predicate.

Definition 9.7 *(Liu [84]) Uncertain predicate is an uncertain set representing a property that the individuals have in common.*

Example 9.25: "Today is warm" is a statement in which "warm" is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/3, & \text{if } 15 \le x \le 18 \\ 1, & \text{if } 18 \le x \le 24 \\ (28-x)/4, & \text{if } 24 \le x \le 28 \\ 0, & \text{if } 28 \le x. \end{cases} \tag{9.45}$$

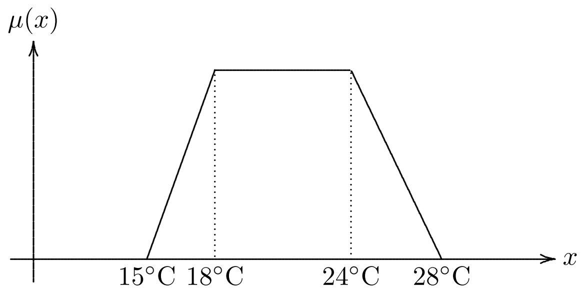


Figure 9.12: Membership Function of Predicate "*warm*"

Example 9.26: "John is young" is a statement in which "young" is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 35 \\ (45-x)/10, & \text{if } 35 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{9.46}$$

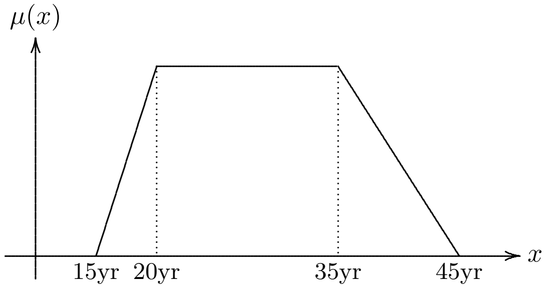


Figure 9.13: Membership Function of Predicate "young"

Example 9.27: "Tom is tall" is a statement in which "tall" is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x-180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } 185 \le x \le 195 \\ (200-x)/5, & \text{if } 195 \le x \le 200 \\ 0, & \text{if } x \ge 200. \end{cases} \tag{9.47}$$

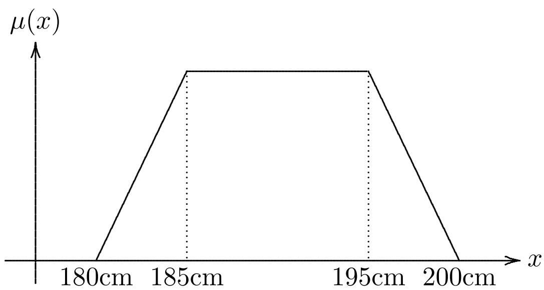


Figure 9.14: Membership Function of Predicate "tall"

Negated Predicate

Definition 9.8 *(Liu [84]) Let P be an uncertain predicate. Then its negated predicate $\neg P$ is the complement of P in the sense of uncertain set, i.e.,*

$$\neg P = P^c. \tag{9.48}$$

Theorem 9.8 *Let P be an uncertain predicate with membership function μ. Then its negated predicate $\neg P$ has a membership function*

$$\neg\mu(x) = 1 - \mu(x). \tag{9.49}$$

Proof: The theorem follows from the definition of negated predicate and the operational law of uncertain set immediately.

Example 9.28: Let P be the uncertain predicate "warm" defined by (9.45). Then its negated predicate $\neg P$ has a membership function

$$\neg\mu(x) = \begin{cases} 1, & \text{if } x \le 15 \\ (18-x)/3, & \text{if } 15 \le x \le 18 \\ 0, & \text{if } 18 \le x \le 24 \\ (x-24)/4, & \text{if } 24 \le x \le 28 \\ 1, & \text{if } 28 \le x. \end{cases} \tag{9.50}$$

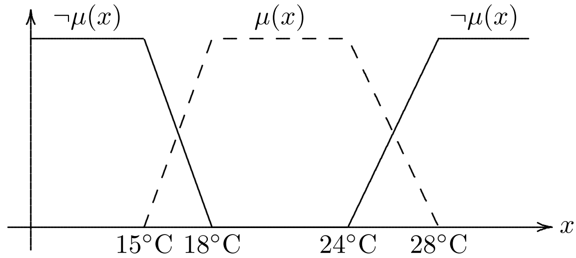


Figure 9.15: Membership Function of Negated Predicate of "*warm*"

Example 9.29: Let P be the uncertain predicate "*young*" defined by (9.46). Then its negated predicate $\neg P$ has a membership function

$$\neg\mu(x) = \begin{cases} 1, & \text{if } x \le 15 \\ (20-x)/5, & \text{if } 15 \le x \le 20 \\ 0, & \text{if } 20 \le x \le 35 \\ (x-35)/10, & \text{if } 35 \le x \le 45 \\ 1, & \text{if } x \ge 45. \end{cases} \tag{9.51}$$

Example 9.30: Let P be the uncertain predicate "*tall*" defined by (9.47). Then its negated predicate $\neg P$ has a membership function

$$\neg\mu(x) = \begin{cases} 1, & \text{if } x \le 180 \\ (185-x)/5, & \text{if } 180 \le x \le 185 \\ 0, & \text{if } 185 \le x \le 195 \\ (x-195)/5, & \text{if } 195 \le x \le 200 \\ 1, & \text{if } x \ge 200. \end{cases} \tag{9.52}$$

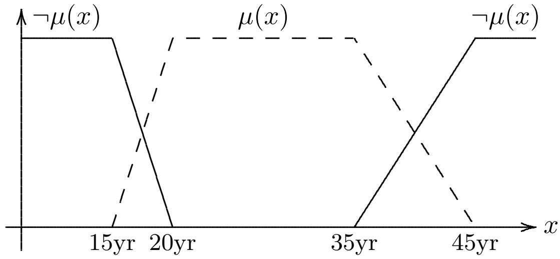


Figure 9.16: Membership Function of Negated Predicate of "*young*"

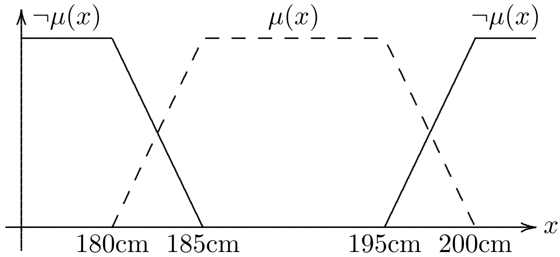


Figure 9.17: Membership Function of Negated Predicate of "*tall*"

Theorem 9.9 *Let P be an uncertain predicate. Then we have $\neg\neg P = P$.*

Proof: The theorem follows from $\neg\neg P = \neg P^c = (P^c)^c = P$.

9.5 Uncertain Proposition

Definition 9.9 *(Liu [84]) Assume that $\mathbb{Q}$ is an uncertain quantifier, S is an uncertain subject, and P is an uncertain predicate. Then the triplet*

$$(\mathbb{Q}, S, P) = \text{"}\mathbb{Q} \textit{ of } S \textit{ are } P\text{"} \tag{9.53}$$

is called an uncertain proposition.

Remark 9.2: Let A be the universe of individuals. Then $(\mathbb{Q}, A, P)$ is a special uncertain proposition because A itself is a special uncertain subject.

Remark 9.3: Let $\forall$ be the universal quantifier. Then $(\forall, A, P)$ is an uncertain proposition representing "all of A are P".

Remark 9.4: Let $\exists$ be the existential quantifier. Then $(\exists, A, P)$ is an uncertain proposition representing "at least one of A is P".

Example 9.31: "Almost all students are young" is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is "almost all", the uncertain subject S is "students" (the universe itself) and the uncertain predicate P is "young".

Example 9.32: "Most young students are tall" is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is "most", the uncertain subject S is "young students" and the uncertain predicate P is "tall".

Theorem 9.10 *(Liu [84], Logical Equivalence Theorem) Let* $(\mathcal{Q}, S, P)$ *be an uncertain proposition. Then*

$$(\mathcal{Q}^*, S, P) = (\mathcal{Q}, S, \neg P) \tag{9.54}$$

where $\mathcal{Q}^*$ *is the dual quantifier of* $\mathcal{Q}$ *and* $\neg P$ *is the negated predicate of* P.

Proof: Note that $(\mathcal{Q}^*, S, P)$ represents "$\mathcal{Q}^*$ of S are P". In fact, the statement "$\mathcal{Q}^*$ of S are P" implies "$\mathcal{Q}^{**}$ of S are not P". Since $\mathcal{Q}^{**} = \mathcal{Q}$, we obtain $(\mathcal{Q}, S, \neg P)$. Conversely, the statement "$\mathcal{Q}$ of S are not P" implies "$\mathcal{Q}^*$ of S are P", i.e., $(\mathcal{Q}^*, S, P)$. Thus (9.54) is verified.

Example 9.33: When $\mathcal{Q}^* = \neg\forall$, we have $\mathcal{Q} = \exists$. If $S = A$, then (9.54) becomes the classical equivalence

$$(\neg\forall, A, P) = (\exists, A, \neg P). \tag{9.55}$$

Example 9.34: When $\mathcal{Q}^* = \neg\exists$, we have $\mathcal{Q} = \forall$. If $S = A$, then (9.54) becomes the classical equivalence

$$(\neg\exists, A, P) = (\forall, A, \neg P). \tag{9.56}$$

9.6 Truth Value

Let $(\mathcal{Q}, S, P)$ be an uncertain proposition. The truth value of $(\mathcal{Q}, S, P)$ should be the uncertain measure that "$\mathcal{Q}$ of S are P". That is,

$$T(\mathcal{Q}, S, P) = \mathcal{M}\{\mathcal{Q} \text{ of } S \text{ are } P\}. \tag{9.57}$$

However, it is impossible for us to deduce the value of $\mathcal{M}\{\mathcal{Q} \text{ of } S \text{ are } P\}$ from the information of $\mathcal{Q}$, S and P within the framework of uncertain set theory. Thus we need an additional formula to compose $\mathcal{Q}$, S and P.

Definition 9.10 *(Liu [84]) Let* $(\mathcal{Q}, S, P)$ *be an uncertain proposition in which* $\mathcal{Q}$ *is a unimodal uncertain quantifier with membership function* λ, S *is an uncertain subject with membership function* ν, *and* P *is an uncertain predicate with membership function* μ. *Then the truth value of* $(\mathcal{Q}, S, P)$ *with respect to the universe* A *is*

$$T(\mathcal{Q}, S, P) = \sup_{0 \le \omega \le 1} \left(\omega \wedge \sup_{K \in \mathbb{K}_\omega} \inf_{a \in K} \mu(a) \wedge \sup_{K \in \mathbb{K}_\omega^*} \inf_{a \in K} \neg\mu(a) \right) \tag{9.58}$$

where

$$\mathbb{K}_\omega = \{K \subset S_\omega \mid \lambda(|K|) \geq \omega\}, \tag{9.59}$$

$$\mathbb{K}^*_\omega = \{K \subset S_\omega \mid \lambda(|S_\omega| - |K|) \geq \omega\}, \tag{9.60}$$

$$S_\omega = \{a \in A \mid \nu(a) \geq \omega\}. \tag{9.61}$$

Remark 9.5: Keep in mind that the truth value formula (9.58) is vacuous if the individual feature data of the universe A are not available.

Remark 9.6: The symbol $|K|$ represents the cardinality of the set K. For example, $|\emptyset| = 0$ and $|\{2, 5, 6\}| = 3$.

Remark 9.7: Note that $\neg\mu$ is the membership function of the negated predicate of P, and

$$\neg\mu(a) = 1 - \mu(a). \tag{9.62}$$

Remark 9.8: When the subset K of individuals becomes an empty set $\emptyset$, we set

$$\inf_{a\in\emptyset} \mu(a) = \inf_{a\in\emptyset} \neg\mu(a) = 1. \tag{9.63}$$

Remark 9.9: If $\mathfrak{Q}$ is an uncertain percentage rather than an absolute quantity, then

$$\mathbb{K}_\omega = \left\{K \subset S_\omega \;\middle|\; \lambda\left(\frac{|K|}{|S_\omega|}\right) \geq \omega\right\}, \tag{9.64}$$

$$\mathbb{K}^*_\omega = \left\{K \subset S_\omega \;\middle|\; \lambda\left(1 - \frac{|K|}{|S_\omega|}\right) \geq \omega\right\}. \tag{9.65}$$

Remark 9.10: If the uncertain subject S is identical to the universe A itself (i.e., $S = A$), then

$$\mathbb{K}_\omega = \{K \subset A \mid \lambda(|K|) \geq \omega\}, \tag{9.66}$$

$$\mathbb{K}^*_\omega = \{K \subset A \mid \lambda(|A| - |K|) \geq \omega\}. \tag{9.67}$$

Exercise 9.1: If the uncertain quantifier $\mathfrak{Q} = \forall$ and the uncertain subject $S = A$, then for any $\omega > 0$, we have

$$\mathbb{K}_\omega = \{A\}, \quad \mathbb{K}^*_\omega = \{\emptyset\}. \tag{9.68}$$

Show that

$$T(\forall, A, P) = \inf_{a\in A} \mu(a). \tag{9.69}$$

Exercise 9.2: If the uncertain quantifier $\mathfrak{Q} = \exists$ and the uncertain subject $S = A$, then for any $\omega > 0$, we have

$$\mathbb{K}_\omega = \{\text{any nonempty subsets of } A\}, \tag{9.70}$$

$$\mathbb{K}^*_\omega = \{\text{any proper subsets of } A\}. \tag{9.71}$$

Show that

$$T(\exists, A, P) = \sup_{a\in A} \mu(a). \tag{9.72}$$

Exercise 9.3: If the uncertain quantifier $\mathcal{Q} = \neg\forall$ and the uncertain subject $S = A$, then for any $\omega > 0$, we have

$$\mathbb{K}_\omega = \{\text{any proper subsets of } A\}, \tag{9.73}$$

$$\mathbb{K}^*_\omega = \{\text{any nonempty subsets of } A\}. \tag{9.74}$$

Show that

$$T(\neg\forall, A, P) = 1 - \inf_{a\in A} \mu(a). \tag{9.75}$$

Exercise 9.4: If the uncertain quantifier $\mathcal{Q} = \neg\exists$ and the uncertain subject $S = A$, then for any $\omega > 0$, we have

$$\mathbb{K}_\omega = \{\emptyset\}, \quad \mathbb{K}^*_\omega = \{A\}. \tag{9.76}$$

Show that

$$T(\neg\exists, A, P) = 1 - \sup_{a\in A} \mu(a). \tag{9.77}$$

Theorem 9.11 *(Liu [84], Truth Value Theorem) Let $(\mathcal{Q}, S, P)$ be an uncertain proposition in which $\mathcal{Q}$ is a unimodal uncertain quantifier with membership function λ, S is an uncertain subject with membership function ν, and P is an uncertain predicate with membership function μ. Then the truth value of $(\mathcal{Q}, S, P)$ is*

$$T(\mathcal{Q}, S, P) = \sup_{0\le\omega\le 1} (\omega \wedge \Delta(k_\omega) \wedge \Delta^*(k^*_\omega)) \tag{9.78}$$

where

$$k_\omega = \min\{x \mid \lambda(x) \ge \omega\}, \tag{9.79}$$

$$\Delta(k_\omega) = k_\omega\text{-}\max\{\mu(a_i) \mid a_i \in S_\omega\}, \tag{9.80}$$

$$k^*_\omega = |S_\omega| - \max\{x \mid \lambda(x) \ge \omega\}, \tag{9.81}$$

$$\Delta^*(k^*_\omega) = k^*_\omega\text{-}\max\{1 - \mu(a_i) \mid a_i \in S_\omega\}. \tag{9.82}$$

Proof: Since the supremum is achieved at the subset with minimum cardinality, we have

$$\sup_{K\in\mathbb{K}_\omega} \inf_{a\in K} \mu(a) = \sup_{K\subset S_\omega, |K|=k_\omega} \inf_{a\in K} \mu(a) = \Delta(k_\omega),$$

$$\sup_{K\in\mathbb{K}^*_\omega} \inf_{a\in K} \neg\mu(a) = \sup_{K\subset S_\omega, |K|=k^*_\omega} \inf_{a\in K} \neg\mu(a) = \Delta^*(k^*_\omega).$$

The theorem is thus verified. Please note that $\Delta(0) = \Delta^*(0) = 1$.

Remark 9.11: If $\mathcal{Q}$ is an uncertain percentage rather than an absolute quantity, then

$$k_\omega = \min\left\{x \mid \lambda\left(\frac{x}{|S_\omega|}\right) \geq \omega\right\}, \tag{9.83}$$

$$k_\omega^* = |S_\omega| - \max\left\{x \mid \lambda\left(\frac{x}{|S_\omega|}\right) \geq \omega\right\}. \tag{9.84}$$

Remark 9.12: If the uncertain subject S is identical to the universe A itself (i.e., $S = A$), then

$$k_\omega = \min\{x \mid \lambda(x) \geq \omega\}, \tag{9.85}$$

$$\Delta(k_\omega) = k_\omega\text{-max}\{\mu(a_1), \mu(a_2), \cdots, \mu(a_n)\}, \tag{9.86}$$

$$k_\omega^* = n - \max\{x \mid \lambda(x) \geq \omega\}, \tag{9.87}$$

$$\Delta^*(k_\omega^*) = k_\omega^*\text{-max}\{1 - \mu(a_1), 1 - \mu(a_2), \cdots, 1 - \mu(a_n)\}. \tag{9.88}$$

Exercise 9.5: If the uncertain quantifier $\mathcal{Q} = \{m, m+1, \cdots, n\}$ (i.e., "*there exist at least* m") with $m \geq 1$, then we have $k_\omega = m$ and $k_\omega^* = 0$. Show that

$$T(\mathcal{Q}, A, P) = m\text{-max}\{\mu(a_1), \mu(a_2), \cdots, \mu(a_n)\}. \tag{9.89}$$

Exercise 9.6: If the uncertain quantifier $\mathcal{Q} = \{0, 1, 2, \ldots, m\}$ (i.e., "*there exist at most* m") with $m < n$, then we have $k_\omega = 0$ and $k_\omega^* = n - m$. Show that

$$T(\mathcal{Q}, A, P) = (n - m)\text{-max}\{1 - \mu(a_1), 1 - \mu(a_2), \cdots, 1 - \mu(a_n)\}. \tag{9.90}$$

Example 9.35: Assume that the daily temperatures of some week from Monday to Sunday are

$$22,\ 23,\ 25,\ 28,\ 30,\ 32,\ 36 \tag{9.91}$$

in centigrades. Consider an uncertain proposition

$$(\mathcal{Q}, A, P) = \text{“two or three days are warm”}. \tag{9.92}$$

Note that the uncertain quantifier is $\mathcal{Q} = \{2, 3\}$. We also suppose that the uncertain predicate P = "warm" has a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \leq 15 \\ (x - 15)/3, & \text{if } 15 \leq x \leq 18 \\ 1, & \text{if } 18 \leq x \leq 24 \\ (28 - x)/4, & \text{if } 24 \leq x \leq 28 \\ 0, & \text{if } 28 \leq x. \end{cases} \tag{9.93}$$

It is clear that Monday and Tuesday are warm with truth value 1, and Wednesday is warm with truth value 0.75. But Thursday to Sunday are not "warm" at all (in fact, they are "hot"). Intuitively, the uncertain proposition "two or three days are warm" should be completely true. The truth value formula (9.58) yields that the truth value is

$$T(\text{"two or three days are warm"}) = 1. \tag{9.94}$$

This is an intuitively expected result. In addition, we also have

$$T(\text{"two days are warm"}) = 0.25, \tag{9.95}$$

$$T(\text{"three days are warm"}) = 0.75. \tag{9.96}$$

Example 9.36: Assume that in a class there are 15 students whose ages are

$$21, 22, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35, 36, 38, 40 \tag{9.97}$$

in years. Consider an uncertain proposition

$$(\mathcal{Q}, A, P) = \text{"almost all students are young"}. \tag{9.98}$$

Suppose the uncertain quantifier $\mathcal{Q}$ = "almost all" has a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 10 \\ (x-10)/3, & \text{if } 10 \le x \le 13 \\ 1, & \text{if } 13 \le x \le 15, \end{cases} \tag{9.99}$$

and the uncertain predicate P = "young" has a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 35 \\ (45-x)/10, & \text{if } 35 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{9.100}$$

The truth value formula (9.58) yields that the uncertain proposition has a truth value

$$T(\text{"almost all students are young"}) = 0.9. \tag{9.101}$$

Example 9.37: Assume that in a team there are 16 sportsmen whose heights are

$$\begin{array}{c} 175, 178, 178, 180, 183, 184, 186, 186 \\ 188, 190, 192, 192, 193, 194, 195, 196 \end{array} \tag{9.102}$$

in centimeters. Consider an uncertain proposition

$$(\mathcal{Q}, A, P) = \text{"about 70\% of sportsmen are tall"}. \tag{9.103}$$

Suppose the uncertain quantifier $\mathcal{Q}$ = "about 70%" has a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.6 \\ 20(x - 0.6), & \text{if } 0.6 \le x \le 0.65 \\ 1, & \text{if } 0.65 \le x \le 0.75 \\ 20(0.8 - x), & \text{if } 0.75 \le x \le 0.8 \\ 0, & \text{if } 0.8 \le x \le 1 \end{cases} \tag{9.104}$$

and the uncertain predicate P = "tall" has a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x - 180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } 185 \le x \le 195 \\ (200 - x)/5, & \text{if } 195 \le x \le 200 \\ 0, & \text{if } x \ge 200. \end{cases} \tag{9.105}$$

The truth value formula (9.58) yields that the uncertain proposition has a truth value

$$T(\text{“about 70\% of sportsmen are tall”}) = 0.8. \tag{9.106}$$

Example 9.38: Assume that in a class there are 18 students whose ages and heights are

$$\begin{array}{l} (24, 185),\ (25, 190),\ (26, 184),\ (26, 170),\ (27, 187),\ (27, 188) \\ (28, 160),\ (30, 190),\ (32, 185),\ (33, 176),\ (35, 185),\ (36, 188) \\ (38, 164),\ (38, 178),\ (39, 182),\ (40, 186),\ (42, 165),\ (44, 170) \end{array} \tag{9.107}$$

in years and centimeters. Consider an uncertain proposition

$$(\mathcal{Q}, S, P) = \text{“most young students are tall”}. \tag{9.108}$$

Suppose the uncertain quantifier (percentage) $\mathcal{Q}$ = "most" has a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.7 \\ 20(x - 0.7), & \text{if } 0.7 \le x \le 0.75 \\ 1, & \text{if } 0.75 \le x \le 0.85 \\ 20(0.9 - x), & \text{if } 0.85 \le x \le 0.9 \\ 0, & \text{if } 0.9 \le x \le 1. \end{cases} \tag{9.109}$$

Note that each individual is described by a feature data (y, z), where y represents ages and z represents heights. In this case, the uncertain subject

S = "young students" has a membership function

$$\nu(y) = \begin{cases} 0, & \text{if } y \le 15 \\ (y-15)/5, & \text{if } 15 \le y \le 20 \\ 1, & \text{if } 20 \le y \le 35 \\ (45-y)/10, & \text{if } 35 \le y \le 45 \\ 0, & \text{if } y \ge 45 \end{cases} \tag{9.110}$$

and the uncertain predicate P = "tall" has a membership function

$$\mu(z) = \begin{cases} 0, & \text{if } z \le 180 \\ (z-180)/5, & \text{if } 180 \le z \le 185 \\ 1, & \text{if } 185 \le z \le 195 \\ (200-z)/5, & \text{if } 195 \le z \le 200 \\ 0, & \text{if } z \ge 200. \end{cases} \tag{9.111}$$

The truth value formula (9.58) yields that the uncertain proposition has a truth value

$$T(\text{"most young students are tall"}) = 0.8. \tag{9.112}$$

9.7 Linguistic Summarizer

Linguistic summary is a human language statement that is concise and easy-to-understand by humans. For example, "most young students are tall" is a linguistic summary of students' ages and heights. Thus a linguistic summary is a special uncertain proposition whose uncertain quantifier, uncertain subject and uncertain predicate are linguistic terms. Uncertain logic provides a flexible means that is capable of extracting linguistic summary from a collection of raw data.

What inputs does the uncertain logic need? First, we should have some raw data (i.e., the individual feature data),

$$A = \{a_1, a_2, \cdots, a_n\}. \tag{9.113}$$

Next, we should have some linguistic terms to represent quantifiers, for example, "most" and "all". Denote them by a collection of uncertain quantifiers,

$$\mathbb{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_m\}. \tag{9.114}$$

Then, we should have some linguistic terms to represent subjects, for example, "young students" and "old students". Denote them by a collection of uncertain subjects,

$$\mathbb{S} = \{S_1, S_2, \cdots, S_n\}. \tag{9.115}$$

Last, we should have some linguistic terms to represent predicates, for example, "short" and "tall". Denote them by a collection of uncertain predicates,

$$\mathbb{P} = \{P_1, P_2, \cdots, P_k\}. \tag{9.116}$$

One problem of data mining is to choose an uncertain quantifier $\mathcal{Q} \in \mathbb{Q}$, an uncertain subject $S \in \mathbb{S}$ and an uncertain predicate $P \in \mathbb{P}$ such that the truth value of the linguistic summary "$\mathcal{Q}$ of S are P" to be extracted is at least β, i.e.,

$$T(\mathcal{Q}, S, P) \geq \beta \tag{9.117}$$

for the universe $A = \{a_1, a_2, \cdots, a_n\}$, where β is a confidence level. In order to solve this problem, Liu [84] proposed the following linguistic summarizer,

$$\begin{cases} \text{Find } \mathcal{Q}, S \text{ and } P \\ \text{subject to:} \\ \qquad \mathcal{Q} \in \mathbb{Q} \\ \qquad S \in \mathbb{S} \\ \qquad P \in \mathbb{P} \\ \qquad T(\mathcal{Q}, S, P) \geq \beta. \end{cases} \tag{9.118}$$

Each solution $(\overline{\mathcal{Q}}, \overline{S}, \overline{P})$ of the linguistic summarizer (9.118) produces a linguistic summary "$\overline{\mathcal{Q}}$ of $\overline{S}$ are $\overline{P}$".

Example 9.39: Assume that in a class there are 18 students whose ages and heights are

$$\begin{array}{l} (24,185),\ (25,190),\ (26,184),\ (26,170),\ (27,187),\ (27,188) \\ (28,160),\ (30,190),\ (32,185),\ (33,176),\ (35,185),\ (36,188) \\ (38,164),\ (38,178),\ (39,182),\ (40,186),\ (42,165),\ (44,170) \end{array} \tag{9.119}$$

in years and centimeters. Suppose we have three linguistic terms "about half", "most" and "all" as uncertain quantifiers whose membership functions are

$$\lambda_{half}(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 0.4 \\ 20(x-0.4), & \text{if } 0.4 \leq x \leq 0.45 \\ 1, & \text{if } 0.45 \leq x \leq 0.55 \\ 20(0.6-x), & \text{if } 0.55 \leq x \leq 0.6 \\ 0, & \text{if } 0.6 \leq x \leq 1, \end{cases} \tag{9.120}$$

$$\lambda_{most}(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 0.7 \\ 20(x-0.7), & \text{if } 0.7 \leq x \leq 0.75 \\ 1, & \text{if } 0.75 \leq x \leq 0.85 \\ 20(0.9-x), & \text{if } 0.85 \leq x \leq 0.9 \\ 0, & \text{if } 0.9 \leq x \leq 1, \end{cases} \tag{9.121}$$

$$\lambda_{all}(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } 0 \le x < 1, \end{cases} \tag{9.122}$$

respectively. Denote the collection of uncertain quantifiers by

$$\mathbb{Q} = \{\text{“about half”}, \text{ “most”}, \text{“all”}\}. \tag{9.123}$$

We also have three linguistic terms “young students”, “middle-aged students” and “old students” as uncertain subjects whose membership functions are

$$\nu_{young}(y) = \begin{cases} 0, & \text{if } y \le 15 \\ (y-15)/5, & \text{if } 15 \le y \le 20 \\ 1, & \text{if } 20 \le y \le 35 \\ (45-y)/10, & \text{if } 35 \le y \le 45 \\ 0, & \text{if } y \ge 45, \end{cases} \tag{9.124}$$

$$\nu_{middle}(y) = \begin{cases} 0, & \text{if } y \le 40 \\ (y-40)/5, & \text{if } 40 \le y \le 45 \\ 1, & \text{if } 45 \le y \le 55 \\ (60-y)/5, & \text{if } 55 \le y \le 60 \\ 0, & \text{if } y \ge 60, \end{cases} \tag{9.125}$$

$$\nu_{old}(y) = \begin{cases} 0, & \text{if } y \le 55 \\ (y-55)/5, & \text{if } 55 \le y \le 60 \\ 1, & \text{if } 60 \le y \le 80 \\ (85-y)/5, & \text{if } 80 \le y \le 85 \\ 1, & \text{if } y \ge 85, \end{cases} \tag{9.126}$$

respectively. Denote the collection of uncertain subjects by

$$\mathbb{S} = \{\text{“young students”}, \text{ “middle-aged students”}, \text{ “old students”}\}. \tag{9.127}$$

Finally, we suppose that there are two linguistic terms “short” and “tall” as uncertain predicates whose membership functions are

$$\mu_{short}(z) = \begin{cases} 0, & \text{if } z \le 145 \\ (z-145)/5, & \text{if } 145 \le z \le 150 \\ 1, & \text{if } 150 \le z \le 155 \\ (160-z)/5, & \text{if } 155 \le z \le 160 \\ 0, & \text{if } z \ge 200, \end{cases} \tag{9.128}$$

$$\mu_{tall}(z) = \begin{cases} 0, & \text{if } z \le 180 \\ (z-180)/5, & \text{if } 180 \le z \le 185 \\ 1, & \text{if } 185 \le z \le 195 \\ (200-z)/5, & \text{if } 195 \le z \le 200 \\ 0, & \text{if } z \ge 200, \end{cases} \tag{9.129}$$

respectively. Denote the collection of uncertain predicates by

$$\mathbb{P} = \{\text{“short”},\ \text{“tall”}\}. \tag{9.130}$$

We would like to extract an uncertain quantifier $\mathcal{Q} \in \mathbb{Q}$, an uncertain subject $S \in \mathbb{S}$ and an uncertain predicate $P \in \mathbb{P}$ such that the truth value of the linguistic summary "$\mathcal{Q}$ of S are P" to be extracted is at least 0.8, i.e.,

$$T(\mathcal{Q}, S, P) \geq 0.8 \tag{9.131}$$

where 0.8 is a predetermined confidence level. The linguistic summarizer (9.118) yields

$$\overline{\mathcal{Q}} = \text{“most”}, \quad \overline{S} = \text{“young students”}, \quad \overline{P} = \text{“tall”}$$

and then extracts a linguistic summary "most young students are tall".

9.8 Bibliographic Notes

Based on uncertain set theory, uncertain logic was designed by Liu [84] in 2011 for dealing with human language by using the truth value formula for uncertain propositions. As an application of uncertain logic, Liu [84] also proposed a linguistic summarizer that provides a means for extracting linguistic summary from a collection of raw data.

Chapter 10

Uncertain Inference

Uncertain inference is a process of deriving consequences from human knowledge via uncertain set theory. This chapter will introduce a family of uncertain inference rules, uncertain system, and uncertain control with application to an inverted pendulum system.

10.1 Uncertain Inference Rule

Let $\mathbb{X}$ and $\mathbb{Y}$ be two concepts. It is assumed that we only have a single if-then rule,

$$\text{“if } \mathbb{X} \text{ is } \xi \text{ then } \mathbb{Y} \text{ is } \eta\text{”} \tag{10.1}$$

where ξ and η are two uncertain sets. We first introduce the following inference rule.

Inference Rule 10.1 *(Liu [81]) Let $\mathbb{X}$ and $\mathbb{Y}$ be two concepts. Assume a rule “if $\mathbb{X}$ is an uncertain set ξ then $\mathbb{Y}$ is an uncertain set η”. From $\mathbb{X}$ is a constant a we infer that $\mathbb{Y}$ is an uncertain set*

$$\eta^* = \eta|_{a\in\xi} \tag{10.2}$$

which is the conditional uncertain set η given $a \in \xi$. The inference rule is represented by

$$\begin{array}{l} \text{Rule: If } \mathbb{X} \text{ is } \xi \text{ then } \mathbb{Y} \text{ is } \eta \\ \text{From: } \mathbb{X} \text{ is a constant } a \\ \hline \text{Infer: } \mathbb{Y} \text{ is } \eta^* = \eta|_{a\in\xi} \end{array} \tag{10.3}$$

Theorem 10.1 *(Liu [81]) In Inference Rule 10.1, if ξ and η are independent uncertain sets with membership functions μ and ν, respectively, then η^* has* *a membership function*

$$\nu^*(y) = \begin{cases} \dfrac{\nu(y)}{\mu(a)}, & \text{if } \nu(y) < \mu(a)/2 \\ \dfrac{\nu(y)+\mu(a)-1}{\mu(a)}, & \text{if } \nu(y) > 1-\mu(a)/2 \\ 0.5, & \text{otherwise.} \end{cases} \tag{10.4}$$

Proof: It follows from Inference Rule 10.1 that η^* is the conditional uncertain set η given $a \in \xi$. By applying Theorem 8.46, the membership function of η^* is just ν^*.

Inference Rule 10.2 *(Gao-Gao-Ralescu [41]) Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ be three concepts. Assume a rule "if $\mathbb{X}$ is an uncertain set ξ and $\mathbb{Y}$ is an uncertain set η then $\mathbb{Z}$ is an uncertain set τ". From $\mathbb{X}$ is a constant a and $\mathbb{Y}$ is a constant b we infer that $\mathbb{Z}$ is an uncertain set*

$$\tau^* = \tau|_{(a\in\xi)\cap(b\in\eta)} \tag{10.5}$$

which is the conditional uncertain set τ given $a \in \xi$ and $b \in \eta$. The inference rule is represented by

$$\begin{array}{l} \text{Rule: If } \mathbb{X} \text{ is } \xi \text{ and } \mathbb{Y} \text{ is } \eta \text{ then } \mathbb{Z} \text{ is } \tau \\ \text{From: } \mathbb{X} \text{ is } a \text{ and } \mathbb{Y} \text{ is } b \\ \hline \text{Infer: } \mathbb{Z} \text{ is } \tau^* = \tau|_{(a\in\xi)\cap(b\in\eta)} \end{array} \tag{10.6}$$

Theorem 10.2 *(Gao-Gao-Ralescu [41]) In Inference Rule 10.2, if ξ, η, τ are independent uncertain sets with membership functions μ, ν, λ, respectively, then τ^* has a membership function*

$$\lambda^*(z) = \begin{cases} \dfrac{\lambda(z)}{\mu(a)\wedge\nu(b)}, & \text{if } \lambda(z) < \dfrac{\mu(a)\wedge\nu(b)}{2} \\ \dfrac{\lambda(z)+\mu(a)\wedge\nu(b)-1}{\mu(a)\wedge\nu(b)}, & \text{if } \lambda(z) > 1-\dfrac{\mu(a)\wedge\nu(b)}{2} \\ 0.5, & \text{otherwise.} \end{cases} \tag{10.7}$$

Proof: It follows from Inference Rule 10.2 that τ^* is the conditional uncertain set τ given $a \in \xi$ and $b \in \eta$. By applying Theorem 8.46, the membership function of τ^* is just λ^*.

Inference Rule 10.3 *(Gao-Gao-Ralescu [41]) Let $\mathbb{X}$ and $\mathbb{Y}$ be two concepts. Assume two rules "if $\mathbb{X}$ is an uncertain set ξ_1 then $\mathbb{Y}$ is an uncertain set η_1" and "if $\mathbb{X}$ is an uncertain set ξ_2 then $\mathbb{Y}$ is an uncertain set η_2". From $\mathbb{X}$ is a constant a we infer that $\mathbb{Y}$ is an uncertain set*

$$\eta^* = \frac{\mathcal{M}\{a\in\xi_1\}\cdot\eta_1|_{a\in\xi_1}}{\mathcal{M}\{a\in\xi_1\}+\mathcal{M}\{a\in\xi_2\}} + \frac{\mathcal{M}\{a\in\xi_2\}\cdot\eta_2|_{a\in\xi_2}}{\mathcal{M}\{a\in\xi_1\}+\mathcal{M}\{a\in\xi_2\}}. \tag{10.8}$$

The inference rule is represented by

$$
\begin{array}{l}
\text{Rule 1: If } \mathbb{X} \text{ is } \xi_1 \text{ then } \mathbb{Y} \text{ is } \eta_1 \\
\text{Rule 2: If } \mathbb{X} \text{ is } \xi_2 \text{ then } \mathbb{Y} \text{ is } \eta_2 \\
\text{From: } \mathbb{X} \text{ is a constant } a \\
\hline
\text{Infer: } \mathbb{Y} \text{ is } \eta^* \text{ determined by (10.8)}
\end{array}
\tag{10.9}
$$

Theorem 10.3 *(Gao-Gao-Ralescu [41]) In Inference Rule 10.3, if $\xi_1, \xi_2, \eta_1, \eta_2$ are independent uncertain sets with membership functions $\mu_1, \mu_2, \nu_1, \nu_2$, respectively, then*

$$\eta^* = \frac{\mu_1(a)}{\mu_1(a) + \mu_2(a)} \eta_1^* + \frac{\mu_2(a)}{\mu_1(a) + \mu_2(a)} \eta_2^* \tag{10.10}$$

where η_1^* and η_2^* are uncertain sets whose membership functions are respectively given by

$$\nu_1^*(y) = \begin{cases} \dfrac{\nu_1(y)}{\mu_1(a)}, & \text{if } \nu_1(y) < \mu_1(a)/2 \\ \dfrac{\nu_1(y) + \mu_1(a) - 1}{\mu_1(a)}, & \text{if } \nu_1(y) > 1 - \mu_1(a)/2 \\ 0.5, & \text{otherwise,} \end{cases} \tag{10.11}$$

$$\nu_2^*(y) = \begin{cases} \dfrac{\nu_2(y)}{\mu_2(a)}, & \text{if } \nu_2(y) < \mu_2(a)/2 \\ \dfrac{\nu_2(y) + \mu_2(a) - 1}{\mu_2(a)}, & \text{if } \nu_2(y) > 1 - \mu_2(a)/2 \\ 0.5, & \text{otherwise.} \end{cases} \tag{10.12}$$

Proof: It follows from Inference Rule 10.3 that the uncertain set η^* is just

$$\eta^* = \frac{\mathcal{M}\{a \in \xi_1\} \cdot \eta_1|_{a \in \xi_1}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}} + \frac{\mathcal{M}\{a \in \xi_2\} \cdot \eta_2|_{a \in \xi_2}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}}.$$

The theorem follows from $\mathcal{M}\{a \in \xi_1\} = \mu_1(a)$ and $\mathcal{M}\{a \in \xi_2\} = \mu_2(a)$ immediately.

Inference Rule 10.4 *(Gao-Gao-Ralescu [41]) Let $\mathbb{X}_1, \mathbb{X}_2, \cdots, \mathbb{X}_m$ be concepts. Assume rules "if $\mathbb{X}_1$ is ξ_{i1} and $\cdots$ and $\mathbb{X}_m$ is ξ_{im} then $\mathbb{Y}$ is η_i" for $i = 1, 2, \cdots, k$. From $\mathbb{X}_1$ is a_1 and $\cdots$ and $\mathbb{X}_m$ is a_m we infer that $\mathbb{Y}$ is an uncertain set*

$$\eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_i|_{(a_1 \in \xi_{i1}) \cap (a_2 \in \xi_{i2}) \cap \cdots \cap (a_m \in \xi_{im})}}{c_1 + c_2 + \cdots + c_k} \tag{10.13}$$

where the coefficients are determined by

$$c_i = \mathcal{M}\{(a_1 \in \xi_{i1}) \cap (a_2 \in \xi_{i2}) \cap \cdots \cap (a_m \in \xi_{im})\} \tag{10.14}$$

for $i = 1, 2, \cdots, k$. The inference rule is represented by

$$\begin{array}{l}
\text{Rule 1: If } \mathbb{X}_1 \text{ is } \xi_{11} \text{ and } \cdots \text{ and } \mathbb{X}_m \text{ is } \xi_{1m} \text{ then } \mathbb{Y} \text{ is } \eta_1 \\
\text{Rule 2: If } \mathbb{X}_1 \text{ is } \xi_{21} \text{ and } \cdots \text{ and } \mathbb{X}_m \text{ is } \xi_{2m} \text{ then } \mathbb{Y} \text{ is } \eta_2 \\
\quad \cdots \\
\text{Rule } k\text{: If } \mathbb{X}_1 \text{ is } \xi_{k1} \text{ and } \cdots \text{ and } \mathbb{X}_m \text{ is } \xi_{km} \text{ then } \mathbb{Y} \text{ is } \eta_k \\
\text{From: } \mathbb{X}_1 \text{ is } a_1 \text{ and } \cdots \text{ and } \mathbb{X}_m \text{ is } a_m \\
\hline
\text{Infer: } \mathbb{Y} \text{ is } \eta^* \text{ determined by (10.13)}
\end{array} \tag{10.15}$$

Theorem 10.4 *(Gao-Gao-Ralescu [41]) In Inference Rule 10.4, if $\xi_{i1}, \xi_{i2}, \cdots, \xi_{im}, \eta_i$ are independent uncertain sets with membership functions $\mu_{i1}, \mu_{i2}, \cdots, \mu_{im}, \nu_i$, $i = 1, 2, \cdots, k$, respectively, then*

$$\eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_i^*}{c_1 + c_2 + \cdots + c_k} \tag{10.16}$$

where η_i^* are uncertain sets whose membership functions are given by

$$\nu_i^*(y) = \begin{cases} \dfrac{\nu_i(y)}{c_i}, & \text{if } \nu_i(y) < c_i/2 \\ \dfrac{\nu_i(y) + c_i - 1}{c_i}, & \text{if } \nu_i(y) > 1 - c_i/2 \\ 0.5, & \text{otherwise} \end{cases} \tag{10.17}$$

and c_i are constants determined by

$$c_i = \min_{1 \le l \le m} \mu_{il}(a_l) \tag{10.18}$$

for $i = 1, 2, \cdots, k$, respectively.

Proof: For each i, since $\{a_1 \in \xi_{i1}\}, \{a_2 \in \xi_{i2}\}, \cdots, \{a_m \in \xi_{im}\}$ are independent events, we immediately have

$$\mathcal{M}\left\{\bigcap_{j=1}^{m} (a_j \in \xi_{ij})\right\} = \min_{1 \le j \le m} \mathcal{M}\{a_j \in \xi_{ij}\} = \min_{1 \le l \le m} \mu_{il}(a_l)$$

for $i = 1, 2, \cdots, k$. From those equations, we may prove the theorem by Inference Rule 10.4 immediately.

10.2 Uncertain System

Uncertain system, proposed by Liu [81], is a function from its inputs to outputs based on the uncertain inference rule. Usually, an uncertain system consists of 5 parts:

1. inputs that are crisp data to be fed into the uncertain system;
2. a rule-base that contains a set of if-then rules provided by the experts;
3. an uncertain inference rule that infers uncertain consequents from the uncertain antecedents;
4. an expected value operator that converts the uncertain consequents to crisp values;
5. outputs that are crisp data yielded from the expected value operator.

Now let us consider an uncertain system in which there are m crisp inputs $\alpha_1, \alpha_2, \cdots, \alpha_m$, and n crisp outputs $\beta_1, \beta_2, \cdots, \beta_n$. At first, we infer n uncertain sets $\eta_1^*, \eta_2^*, \cdots, \eta_n^*$ from the m crisp inputs by the rule-base (i.e., a set of if-then rules),

$$\begin{array}{l} \text{If } \xi_{11} \text{ and } \xi_{12} \text{ and} \cdots \text{and } \xi_{1m} \text{ then } \eta_{11} \text{ and } \eta_{12} \text{ and} \cdots \text{and } \eta_{1n} \\ \text{If } \xi_{21} \text{ and } \xi_{22} \text{ and} \cdots \text{and } \xi_{2m} \text{ then } \eta_{21} \text{ and } \eta_{22} \text{ and} \cdots \text{and } \eta_{2n} \\ \quad \cdots \\ \text{If } \xi_{k1} \text{ and } \xi_{k2} \text{ and} \cdots \text{and } \xi_{km} \text{ then } \eta_{k1} \text{ and } \eta_{k2} \text{ and} \cdots \text{and } \eta_{kn} \end{array} \tag{10.19}$$

and the uncertain inference rule

$$\eta_j^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_{ij}|_{(\alpha_1 \in \xi_{i1}) \cap (\alpha_2 \in \xi_{i2}) \cap \cdots \cap (\alpha_m \in \xi_{im})}}{c_1 + c_2 + \cdots + c_k} \tag{10.20}$$

for $j = 1, 2, \cdots, n$, where the coefficients are determined by

$$c_i = \mathcal{M}\{(\alpha_1 \in \xi_{i1}) \cap (\alpha_2 \in \xi_{i2}) \cap \cdots \cap (\alpha_m \in \xi_{im})\} \tag{10.21}$$

for $i = 1, 2, \cdots, k$. Thus by using the expected value operator, we obtain

$$\beta_j = E[\eta_j^*] \tag{10.22}$$

for $j = 1, 2, \cdots, n$. Until now we have constructed a function from inputs $\alpha_1, \alpha_2, \cdots, \alpha_m$ to outputs $\beta_1, \beta_2, \cdots, \beta_n$. Write this function by f, i.e.,

$$(\beta_1, \beta_2, \cdots, \beta_n) = f(\alpha_1, \alpha_2, \cdots, \alpha_m). \tag{10.23}$$

Then we get an uncertain system f.

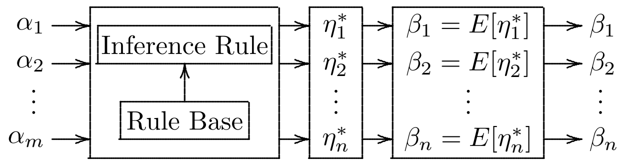


Figure 10.1: An Uncertain System

Theorem 10.5 *Assume* $\xi_{i1}, \xi_{i2}, \cdots, \xi_{im}, \eta_{i1}, \eta_{i2}, \cdots, \eta_{in}$ *are independent uncertain sets with membership functions* $\mu_{i1}, \mu_{i2}, \cdots, \mu_{im}, \nu_{i1}, \nu_{i2}, \cdots, \nu_{in}$, $i = 1, 2, \cdots, k$, *respectively. Then the uncertain system from* $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ *to* $(\beta_1, \beta_2, \cdots, \beta_n)$ *is*

$$\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot E[\eta_{ij}^*]}{c_1 + c_2 + \cdots + c_k} \tag{10.24}$$

for $j = 1, 2, \cdots, n$, *where* η_{ij}^* *are uncertain sets whose membership functions are given by*

$$\nu_{ij}^*(y) = \begin{cases} \dfrac{\nu_{ij}(y)}{c_i}, & \text{if } \nu_{ij}(y) < c_i/2 \\ \dfrac{\nu_{ij}(y) + c_i - 1}{c_i}, & \text{if } \nu_{ij}(y) > 1 - c_i/2 \\ 0.5, & \text{otherwise} \end{cases} \tag{10.25}$$

and c_i *are constants determined by*

$$c_i = \min_{1 \le l \le m} \mu_{il}(\alpha_l) \tag{10.26}$$

for $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, n$, *respectively.*

Proof: It follows from Inference Rule 10.4 that the uncertain sets η_j^* are

$$\eta_j^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_{ij}^*}{c_1 + c_2 + \cdots + c_k}$$

for $j = 1, 2, \cdots, n$. Since $\eta_{ij}^*, i = 1, 2, \cdots, k, j = 1, 2, \cdots, n$ are independent uncertain sets, we get the theorem immediately by the linearity of expected value operator.

Remark 10.1: The uncertain system allows the uncertain sets η_{ij} in the rule-base (10.19) become constants b_{ij}, i.e.,

$$\eta_{ij} = b_{ij} \tag{10.27}$$

for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n$. In this case, the uncertain system (10.24) becomes

$$\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot b_{ij}}{c_1 + c_2 + \cdots + c_k} \tag{10.28}$$

for $j = 1, 2, \cdots, n$.

Remark 10.2: The uncertain system allows the uncertain sets η_{ij} in the rule-base (10.19) become functions h_{ij} of inputs $\alpha_1, \alpha_2, \cdots, \alpha_m$, i.e.,

$$\eta_{ij} = h_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_m) \tag{10.29}$$

for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n$. In this case, the uncertain system (10.24) becomes

$$\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot h_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_m)}{c_1 + c_2 + \cdots + c_k} \tag{10.30}$$

for $j = 1, 2, \cdots, n$.

Uncertain Systems are Universal Approximator

Uncertain systems are capable of approximating any continuous function on a compact set (i.e., bounded and closed set) to arbitrary accuracy. This is the reason why uncertain systems may play a controller. The following theorem shows this fact.

Theorem 10.6 *(Peng-Chen [122]) For any given continuous function g on a compact set $D \subset \Re^m$ and any given $\varepsilon > 0$, there exists an uncertain system f such that*

$$\|f(\alpha_1, \alpha_2, \cdots, \alpha_m) - g(\alpha_1, \alpha_2, \cdots, \alpha_m)\| < \varepsilon \tag{10.31}$$

for any $(\alpha_1, \alpha_2, \cdots, \alpha_m) \in D$.

Proof: Without loss of generality, we assume that the function g is a real-valued function with only two variables α_1 and α_2, and the compact set is a unit rectangle $D = [0, 1] \times [0, 1]$. Since g is continuous on D and then is uniformly continuous, for any given number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$|g(\alpha_1, \alpha_2) - g(\alpha_1', \alpha_2')| < \varepsilon \tag{10.32}$$

whenever $\|(\alpha_1, \alpha_2) - (\alpha_1', \alpha_2')\| < \delta$. Let k be an integer larger than $\sqrt{2}/\delta$, and write

$$D_{ij} = \left\{(\alpha_1, \alpha_2) \mid \frac{i-1}{k} < \alpha_1 \le \frac{i}{k}, \ \frac{j-1}{k} < \alpha_2 \le \frac{j}{k}\right\} \tag{10.33}$$

for $i, j = 1, 2, \cdots, k$. Note that $\{D_{ij}\}$ is a sequence of disjoint rectangles whose "diameter" is less than δ. Define uncertain sets

$$\xi_i = \left(\frac{i-1}{k}, \frac{i}{k}\right), \quad i = 1, 2, \cdots, k, \tag{10.34}$$

$$\eta_j = \left(\frac{j-1}{k}, \frac{j}{k}\right), \quad j = 1, 2, \cdots, k. \tag{10.35}$$

Then we assume a rule-base with $k \times k$ if-then rules,

$$\text{Rule } ij\text{: If } \xi_i \text{ and } \eta_j \text{ then } g(i/k, j/k), \quad i, j = 1, 2, \cdots, k. \tag{10.36}$$

According to the uncertain inference rule, the corresponding uncertain system from D to $\Re$ is

$$f(\alpha_1, \alpha_2) = g(i/k, j/k), \quad \text{if } (\alpha_1, \alpha_2) \in D_{ij},\ i, j = 1, 2, \cdots, k. \tag{10.37}$$

It follows from (10.32) that for any $(\alpha_1, \alpha_2) \in D_{ij} \subset D$, we have

$$|f(\alpha_1, \alpha_2) - g(\alpha_1, \alpha_2)| = |g(i/k, j/k) - g(\alpha_1, \alpha_2)| < \varepsilon. \tag{10.38}$$

The theorem is thus verified. Hence uncertain systems are universal approximators.

10.3 Uncertain Control

Uncertain controller, designed by Liu [81], is a special uncertain system that maps the state variables of a process under control to the action variables. Thus an uncertain controller consists of the same 5 parts of uncertain system: inputs, a rule-base, an uncertain inference rule, an expected value operator, and outputs. The distinguished point is that the inputs of uncertain controller are the state variables of the process under control, and the outputs are the action variables.

Figure 10.2 shows an uncertain control system consisting of an uncertain controller and a process. Note that t represents time, $\alpha_1(t), \alpha_2(t), \cdots, \alpha_m(t)$ are not only the inputs of uncertain controller but also the outputs of process, and $\beta_1(t), \beta_2(t), \cdots, \beta_n(t)$ are not only the outputs of uncertain controller but also the inputs of process.

10.4 Inverted Pendulum

Inverted pendulum system is a nonlinear unstable system that is widely used as a benchmark for testing control algorithms. Many good techniques already exist for balancing inverted pendulum. Among others, Gao [45] successfully balanced an inverted pendulum by the uncertain controller with 5×5 if-then rules.

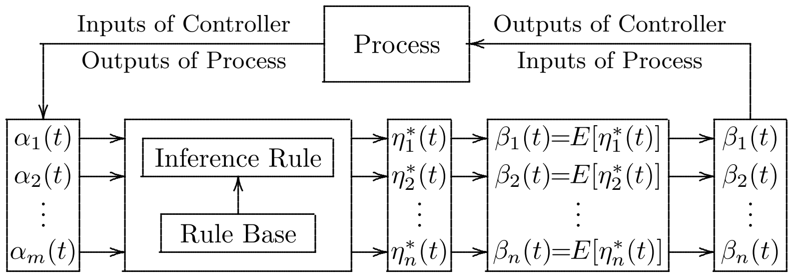


Figure 10.2: An Uncertain Control System

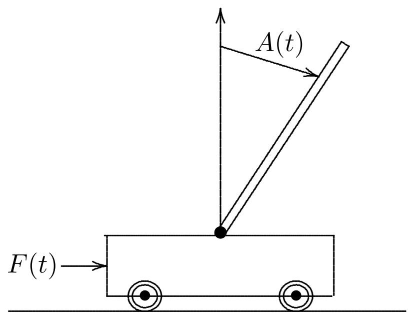


Figure 10.3: An Inverted Pendulum in which $A(t)$ represents the angular position and $F(t)$ represents the force that moves the cart at time t.

The uncertain controller has two inputs ("angle" and "angular velocity") and one output ("force"). Three of them will be represented by uncertain sets labeled by

"negative large"	NL
"negative small"	NS
"zero"	Z
"positive small"	PS
"positive large"	PL

The membership functions of those uncertain sets are shown in Figures 10.4, 10.5 and 10.6.

Intuitively, when the inverted pendulum has a large clockwise angle and a large clockwise angular velocity, we should give it a large force to the right. Thus we have an if-then rule,

If the angle is negative large
and the angular velocity is negative large,
then the force is positive large.

Similarly, when the inverted pendulum has a large counterclockwise angle

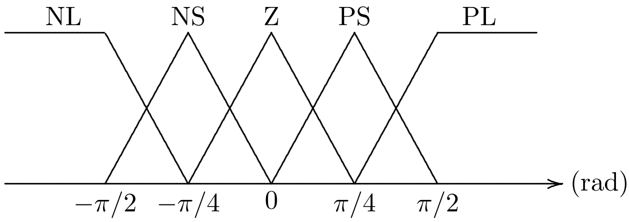


Figure 10.4: Membership Functions of "Angle"

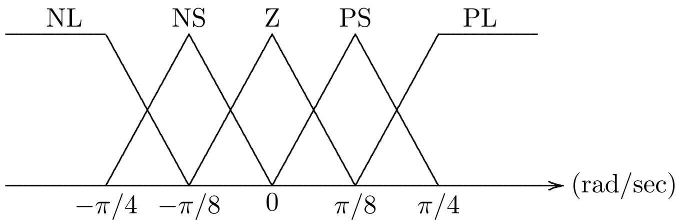


Figure 10.5: Membership Functions of "Angular Velocity"

and a large counterclockwise angular velocity, we should give it a large force to the left. Thus we have an if-then rule,

If the angle is positive large
and the angular velocity is positive large,
then the force is negative large.

Note that each input or output has 5 states and each state is represented by an uncertain set. This implies that the rule-base contains 5×5 if-then rules. In order to balance the inverted pendulum, the 25 if-then rules in Table 10.1 are accepted.

A lot of simulation results show that the uncertain controller may balance the inverted pendulum successfully.

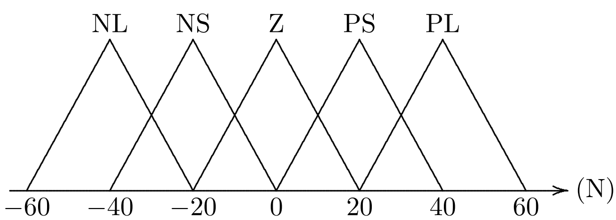


Figure 10.6: Membership Functions of "Force"

Table 10.1: Rule Base with 5 × 5 If-Then Rules

angle \ velocity	NL	NS	Z	PS	PL
NL	PL	PL	PL	PS	Z
NS	PL	PL	PS	Z	NS
Z	PL	PS	Z	NS	NL
PS	PS	Z	NS	NL	NL
PL	Z	NS	NL	NL	NL

10.5 Bibliographic Notes

The basic uncertain inference rule was initialized by Liu [81] in 2010 by the tool of conditional uncertain set. After that, Gao-Gao-Ralescu [41] extended the uncertain inference rule to the case with multiple antecedents and multiple if-then rules.

Based on the uncertain inference rules, Liu [81] suggested the concept of uncertain system, and then presented the tool of uncertain controller. As an important contribution, Peng-Chen [122] proved that uncertain systems are universal approximator and then demonstrated that the uncertain controller is a reasonable tool. As a successful application, Gao [45] balanced an inverted pendulum by using the uncertain controller.

Chapter 11

Uncertain Process

The study of uncertain process was started by Liu [77] in 2008 for modelling the evolution of uncertain phenomena. This chapter will give the concept of uncertain process, and introduce sample path, uncertainty distribution, independent increment process, extreme value, first hitting time, time integral, and stationary increment process.

11.1 Uncertain Process

An uncertain process is essentially a sequence of uncertain variables indexed by time. A formal definition is given below.

Definition 11.1 *(Liu [77]) Let* $(\Gamma,\mathcal{L},\mathcal{M})$ *be an uncertainty space and let* T *be a totally ordered set (e.g. time). An uncertain process is a function* $X_t(\gamma)$ *from* $T\times(\Gamma,\mathcal{L},\mathcal{M})$ *to the set of real numbers such that* $\{X_t\in B\}$ *is an event for any Borel set* B *of real numbers at each time* t.

Remark 11.1: The above definition says X_t is an uncertain process if and only if it is an uncertain variable at each time t.

Example 11.1: Take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\}=0.6$, $\mathcal{M}\{\gamma_2\}=0.4$. Then

$$X_t(\gamma)=\begin{cases} t, & \text{if } \gamma=\gamma_1\\ t+1, & \text{if } \gamma=\gamma_2\end{cases} \tag{11.1}$$

is an uncertain process.

Example 11.2: Take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Then

$$X_t(\gamma)=t-\gamma,\quad \forall\gamma\in\Gamma \tag{11.2}$$

is an uncertain process.

Example 11.3: A real-valued function $f(t)$ with respect to time t may be regarded as a special uncertain process on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, i.e.,

$$X_t(\gamma) = f(t), \quad \forall \gamma \in \Gamma. \tag{11.3}$$

Sample Path

Definition 11.2 *(Liu [77]) Let X_t be an uncertain process. Then for each $\gamma \in \Gamma$, the function $X_t(\gamma)$ is called a sample path of X_t.*

Note that each sample path is a real-valued function of time t. In addition, an uncertain process may also be regarded as a function from an uncertainty space to a collection of sample paths.

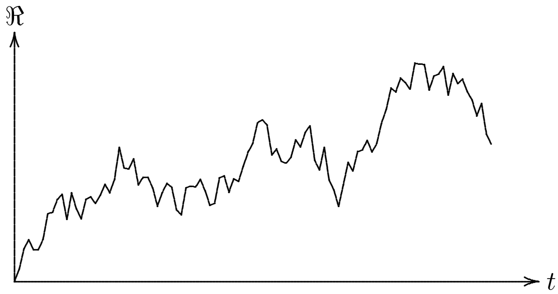


Figure 11.1: A Sample Path of Uncertain Process

Definition 11.3 *An uncertain process X_t is said to be sample-continuous if almost all sample paths are continuous functions with respect to time t.*

11.2 Uncertainty Distribution

An uncertainty distribution of uncertain process is a sequence of uncertainty distributions of uncertain variables indexed by time. Thus an uncertainty distribution of uncertain process is a surface rather than a curve. A formal definition is given below.

Definition 11.4 *(Liu [93]) The uncertainty distribution $\Phi_t(x)$ of an uncertain process X_t is defined by*

$$\Phi_t(x) = \mathcal{M}\{X_t \leq x\} \tag{11.4}$$

for any time t and any number x.

That is, the uncertain process X_t has an uncertainty distribution $\Phi_t(x)$ if at each time t, the uncertain variable X_t has the uncertainty distribution $\Phi_t(x)$.

Example 11.4: The linear uncertain process $X_t \sim \mathcal{L}(at, bt)$ has an uncertainty distribution,

$$\Phi_t(x) = \begin{cases} 0, & \text{if } x \le at \\ \dfrac{x - at}{(b-a)t}, & \text{if } at \le x \le bt \\ 1, & \text{if } x \ge bt. \end{cases} \tag{11.5}$$

Example 11.5: The zigzag uncertain process $X_t \sim \mathcal{Z}(at, bt, ct)$ has an uncertainty distribution,

$$\Phi_t(x) = \begin{cases} 0, & \text{if } x \le at \\ \dfrac{x - at}{2(b-a)t}, & \text{if } at \le x \le bt \\ \dfrac{x + ct - 2bt}{2(c-b)t}, & \text{if } bt \le x \le ct \\ 1, & \text{if } x \ge ct. \end{cases} \tag{11.6}$$

Example 11.6: The normal uncertain process $X_t \sim \mathcal{N}(et, \sigma t)$ has an uncertainty distribution,

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(et - x)}{\sqrt{3}\sigma t}\right)\right)^{-1}. \tag{11.7}$$

Example 11.7: The lognormal uncertain process $X_t \sim \mathcal{LOGN}(et, \sigma t)$ has an uncertainty distribution,

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(et - \ln x)}{\sqrt{3}\sigma t}\right)\right)^{-1}. \tag{11.8}$$

Exercise 11.1: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.6$, $\mathcal{M}\{\gamma_2\} = 0.4$. Derive the uncertainty distribution of the uncertain process

$$X_t(\gamma) = \begin{cases} t, & \text{if } \gamma = \gamma_1 \\ t + 1, & \text{if } \gamma = \gamma_2. \end{cases} \tag{11.9}$$

Exercise 11.2: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Derive the uncertainty distribution of the uncertain process

$$X_t(\gamma) = t - \gamma, \quad \forall \gamma \in \Gamma. \tag{11.10}$$

Exercise 11.3: A real-valued function $f(t)$ with respect to time t is a special uncertain process. What is the uncertainty distribution of $f(t)$?

Theorem 11.1 *(Liu [93], Sufficient and Necessary Condition) A function $\Phi_t(x) : T \times \Re \to [0,1]$ is an uncertainty distribution of uncertain process if and only if at each time t, it is a monotone increasing function with respect to x except $\Phi_t(x) \equiv 0$ and $\Phi_t(x) \equiv 1$.*

Proof: If $\Phi_t(x)$ is an uncertainty distribution of some uncertain process X_t, then at each time t, $\Phi_t(x)$ is the uncertainty distribution of uncertain variable X_t. It follows from Peng-Iwamura theorem that $\Phi_t(x)$ is a monotone increasing function with respect to x and $\Phi_t(x) \not\equiv 0$, $\Phi_t(x) \not\equiv 1$. Conversely, if at each time t, $\Phi_t(x)$ is a monotone increasing function except $\Phi_t(x) \equiv 0$ and $\Phi_t(x) \equiv 1$, it follows from Peng-Iwamura theorem that there exists an uncertain variable ξ_t whose uncertainty distribution is just $\Phi_t(x)$. Define

$$X_t = \xi_t, \quad \forall t \in T.$$

Then X_t is an uncertain process and has the uncertainty distribution $\Phi_t(x)$. The theorem is verified.

Theorem 11.2 *Let X_t be an uncertain process with uncertainty distribution $\Phi_t(x)$, and let $f(x)$ be a continuous function. Then $f(X_t)$ is also an uncertain process. Furthermore, (i) if $f(x)$ is a strictly increasing function, then $f(X_t)$ has an uncertainty distribution*

$$\Psi_t(x) = \Phi_t(f^{-1}(x)); \tag{11.11}$$

and (ii) if $f(x)$ is a strictly decreasing function and $\Phi_t(x)$ is continuous with respect to x, then $f(X_t)$ has an uncertainty distribution

$$\Psi_t(x) = 1 - \Phi_t(f^{-1}(x)). \tag{11.12}$$

Proof: At each time t, since X_t is an uncertain variable, it follows from Theorem 2.1 that $f(X_t)$ is also an uncertain variable. Thus $f(X_t)$ is an uncertain process. The equations (11.11) and (11.12) may be verified by the operational law of uncertain variables immediately.

Example 11.8: Let X_t be an uncertain process with uncertainty distribution $\Phi_t(x)$. Show that the uncertain process $aX_t + b$ has an uncertainty distribution,

$$\Psi_t(x) = \begin{cases} \Phi_t((x-b)/a), & \text{if } a > 0 \\ 1 - \Phi_t((x-b)/a), & \text{if } a < 0. \end{cases} \tag{11.13}$$

Regular Uncertainty Distribution

Definition 11.5 *(Liu [93]) An uncertainty distribution $\Phi_t(x)$ is said to be regular if at each time t, it is a continuous and strictly increasing function with respect to x at which $0 < \Phi_t(x) < 1$, and*

$$\lim_{x \to -\infty} \Phi_t(x) = 0, \quad \lim_{x \to +\infty} \Phi_t(x) = 1. \tag{11.14}$$

It is clear that linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution and lognormal uncertainty distribution of uncertain process are all regular.

Inverse Uncertainty Distribution

Definition 11.6 *(Liu [93]) Let X_t be an uncertain process with regular uncertainty distribution $\Phi_t(x)$. Then the inverse function $\Phi_t^{-1}(\alpha)$ is called the inverse uncertainty distribution of X_t.*

Note that at each time t, the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$ is well defined on the open interval $(0, 1)$. If needed, we may extend the domain to $[0, 1]$ via

$$\Phi_t^{-1}(0) = \lim_{\alpha \downarrow 0} \Phi_t^{-1}(\alpha), \quad \Phi_t^{-1}(1) = \lim_{\alpha \uparrow 1} \Phi_t^{-1}(\alpha). \tag{11.15}$$

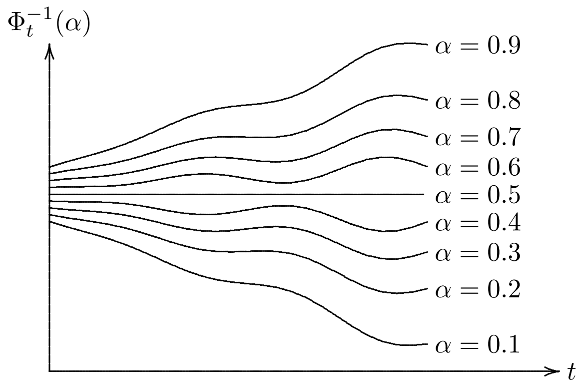


Figure 11.2: Inverse Uncertainty Distribution of Uncertain Process

Example 11.9: The linear uncertain process $X_t \sim \mathcal{L}(at, bt)$ has an inverse uncertainty distribution,

$$\Phi_t^{-1}(\alpha) = (1 - \alpha)at + \alpha bt. \tag{11.16}$$

Example 11.10: The zigzag uncertain process $X_t \sim \mathcal{Z}(at, bt, ct)$ has an inverse uncertainty distribution,

$$\Phi_t^{-1}(\alpha) = \begin{cases} (1-2\alpha)at + 2\alpha bt, & \text{if } \alpha < 0.5 \\ (2-2\alpha)bt + (2\alpha - 1)ct, & \text{if } \alpha \ge 0.5. \end{cases} \tag{11.17}$$

Example 11.11: The normal uncertain process $X_t \sim \mathcal{N}(et, \sigma t)$ has an inverse uncertainty distribution,

$$\Phi_t^{-1}(\alpha) = et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{11.18}$$

Example 11.12: The lognormal uncertain process $X_t \sim \mathcal{LOGN}(et, \sigma t)$ has an inverse uncertainty distribution,

$$\Phi_t^{-1}(\alpha) = \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right). \tag{11.19}$$

Exercise 11.4: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. Derive the inverse uncertainty distribution of the uncertain process

$$X_t(\gamma) = t - \gamma, \quad \forall \gamma \in \Gamma. \tag{11.20}$$

Theorem 11.3 *(Liu [93]) A function $\Phi_t^{-1}(\alpha) : T \times (0,1) \to \Re$ is an inverse uncertainty distribution of uncertain process if at each time t, it is a continuous and strictly increasing function with respect to α.*

Proof: At each time t, since $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to α, it follows from Theorem 2.5 that there exists an uncertain variable ξ_t whose inverse uncertainty distribution is just $\Phi_t^{-1}(\alpha)$. Define

$$X_t = \xi_t, \quad \forall t \in T.$$

Then X_t is an uncertain process and has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. The theorem is proved.

11.3 Independence and Operational Law

Definition 11.7 *(Liu [93]) Uncertain processes $X_{1t}, X_{2t}, \cdots, X_{nt}$ are said to be independent if for any positive integer k and any times $t_1, t_2, \cdots, t_k$, the uncertain vectors*

$$\boldsymbol{\xi}_i = (X_{it_1}, X_{it_2}, \cdots, X_{it_k}), \quad i = 1, 2, \cdots, n \tag{11.21}$$

are independent, i.e., for any Borel sets $B_1, B_2, \cdots, B_n$ of k-dimensional real vectors, we have

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\}. \tag{11.22}$$

Exercise 11.5: Let $X_{1t}, X_{2t}, \cdots, X_{nt}$ be independent uncertain processes, and let $t_1, t_2, \cdots, t_n$ be any times. Show that

$$X_{1t_1}, X_{2t_2}, \cdots, X_{nt_n} \tag{11.23}$$

are independent uncertain variables.

Exercise 11.6: Let X_t and Y_t be independent uncertain processes. For any times $t_1, t_2, \cdots, t_k$ and $s_1, s_2, \cdots, s_m$, show that

$$(X_{t_1}, X_{t_2}, \cdots, X_{t_k}) \text{ and } (Y_{s_1}, Y_{s_2}, \cdots, Y_{s_m}) \tag{11.24}$$

are independent uncertain vectors.

Theorem 11.4 *(Liu [93]) Uncertain processes $X_{1t}, X_{2t}, \cdots, X_{nt}$ are independent if and only if for any positive integer k, any times $t_1, t_2, \cdots, t_k$, and any Borel sets $B_1, B_2, \cdots, B_n$ of k-dimensional real vectors, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\} \tag{11.25}$$

where $\boldsymbol{\xi}_i = (X_{it_1}, X_{it_2}, \cdots, X_{it_k})$ for $i = 1, 2, \cdots, n$.

Proof: It follows from Theorem 2.59 that $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are independent uncertain vectors if and only if (11.25) holds. The theorem is thus verified.

Theorem 11.5 *(Liu [93], Operational Law) Let $X_{1t}, X_{2t}, \cdots, X_{nt}$ be independent uncertain processes with regular uncertainty distributions $\Phi_{1t}, \Phi_{2t}, \cdots, \Phi_{nt}$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then*

$$X_t = f(X_{1t}, X_{2t}, \cdots, X_{nt}) \tag{11.26}$$

has an inverse uncertainty distribution

$$\Phi_t^{-1}(\alpha) = f(\Phi_{1t}^{-1}(\alpha), \cdots, \Phi_{mt}^{-1}(\alpha), \Phi_{m+1,t}^{-1}(1-\alpha), \cdots, \Phi_{nt}^{-1}(1-\alpha)). \tag{11.27}$$

Proof: At any time t, it is clear that $X_{1t}, X_{2t}, \cdots, X_{nt}$ are independent uncertain variables with inverse uncertainty distributions $\Phi_{1t}^{-1}(\alpha), \Phi_{2t}^{-1}(\alpha), \cdots, \Phi_{nt}^{-1}(\alpha)$, respectively. The theorem follows from the operational law of uncertain variables immediately.

Theorem 11.6 *(Operational Law) Let $X_{1t}, X_{2t}, \cdots, X_{nt}$ be independent uncertain processes with continuous uncertainty distributions $\Phi_{1t}, \Phi_{2t}, \cdots, \Phi_{nt}$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then*

$$X_t = f(X_{1t}, X_{2t}, \cdots, X_{nt}) \tag{11.28}$$

has an uncertainty distribution

$$\Phi_t(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \left(\min_{1 \le i \le m} \Phi_{it}(x_i) \wedge \min_{m+1 \le i \le n} (1 - \Phi_{it}(x_i)) \right). \tag{11.29}$$

Proof: At any time t, it is clear that $X_{1t}, X_{2t}, \cdots, X_{nt}$ are independent uncertain variables. The theorem follows from the operational law of uncertain variables immediately.

11.4 Independent Increment Process

An independent increment process is an uncertain process that has independent increments. A formal definition is given below.

Definition 11.8 *(Liu [77]) An uncertain process X_t is said to have independent increments if*

$$X_{t_1}, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots, X_{t_k} - X_{t_{k-1}} \tag{11.30}$$

are independent uncertain variables where $t_1, t_2, \cdots, t_k$ are any times with $t_1 < t_2 < \cdots < t_k$.

That is, an independent increment process means that its increments are independent uncertain variables whenever the time intervals do not overlap. Please note that the increments are also independent of the initial state.

Theorem 11.7 *(Liu [93]) Let $\Phi_t^{-1}(\alpha)$ be the inverse uncertainty distribution of an independent increment process. Then (i) $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to α at each time t, and (ii) $\Phi_t^{-1}(\alpha) - \Phi_s^{-1}(\alpha)$ is a monotone increasing function with respect to α for any times $s < t$.*

Proof: Since $\Phi_t^{-1}(\alpha)$ is the inverse uncertainty distribution of independent increment process X_t, it follows from Theorem 11.3 that $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to α. Since $X_t = X_s + (X_t - X_s)$, for any $\alpha < \beta$, we immediately have

$$\Phi_t^{-1}(\beta) - \Phi_t^{-1}(\alpha) \ge \Phi_s^{-1}(\beta) - \Phi_s^{-1}(\alpha).$$

That is,

$$\Phi_t^{-1}(\beta) - \Phi_s^{-1}(\beta) \ge \Phi_t^{-1}(\alpha) - \Phi_s^{-1}(\alpha).$$

Hence $\Phi_t^{-1}(\alpha)-\Phi_s^{-1}(\alpha)$ is a monotone increasing function with respect to α. The theorem is verified.

Remark 11.2: It follows from Theorem 11.7 that the uncertainty distribution of independent increment process has a horn-like shape. See Figure 11.3.

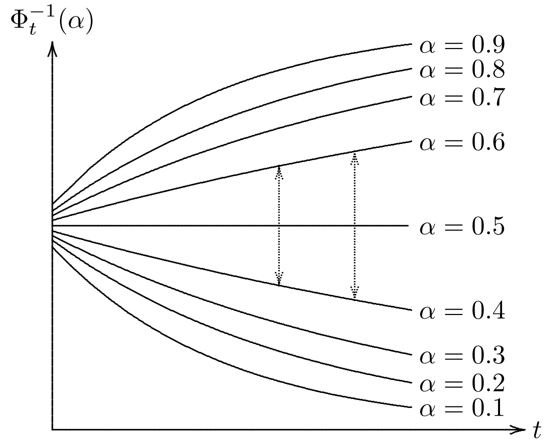


Figure 11.3: Inverse Uncertainty Distribution of Independent Increment Process: A Horn-like Family of Functions of t indexed by α

Theorem 11.8 *(Liu [93]) Let $\Phi_t^{-1}(\alpha): T\times(0,1)\to\Re$ be a function. If (i) $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to α at each time t, and (ii) $\Phi_t^{-1}(\alpha)-\Phi_s^{-1}(\alpha)$ is a monotone increasing function with respect to α for any times $s<t$, then there exists an independent increment process whose inverse uncertainty distribution is just $\Phi_t^{-1}(\alpha)$.*

Proof: Without loss of generality, we only consider the range of $t\in[0,1]$. Let n be a positive integer. Since $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function and $\Phi_t^{-1}(\alpha)-\Phi_s^{-1}(\alpha)$ is a monotone increasing function with respect to α, there exist independent uncertain variables $\xi_{0n},\xi_{1n},\cdots,\xi_{nn}$ such that ξ_{0n} has an inverse uncertainty distribution

$$\Upsilon_{0n}^{-1}(\alpha)=\Phi_0^{-1}(\alpha)$$

and ξ_{in} have uncertainty distributions

$$\Upsilon_{in}(x)=\sup\left\{\alpha\,|\,\Phi_{i/n}^{-1}(\alpha)-\Phi_{(i-1)/n}^{-1}(\alpha)=x\right\},$$

$i=1,2,\cdots,n$, respectively. Define an uncertain process

$$X_t^n=\begin{cases}\displaystyle\sum_{i=0}^{k}\xi_{in}, & \text{if } t=\dfrac{k}{n}\quad(k=0,1,\cdots,n)\\ \text{linear}, & \text{otherwise.}\end{cases}$$

It may prove that X_t^n converges in distribution as $n \to \infty$. Furthermore, we may verify that the limit is indeed an independent increment process and has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. The theorem is verified.

Theorem 11.9 *Let X_t be a sample-continuous independent increment process with regular uncertainty distribution $\Phi_t(x)$. Then for any $\alpha \in (0,1)$, we have*

$$\mathcal{M}\{X_t \le \Phi_t^{-1}(\alpha), \forall t\} = \alpha, \tag{11.31}$$

$$\mathcal{M}\{X_t > \Phi_t^{-1}(\alpha), \forall t\} = 1 - \alpha. \tag{11.32}$$

Proof: It is still a conjecture.

Remark 11.3: It is also showed that for any $\alpha \in (0,1)$, the following two equations are true,

$$\mathcal{M}\{X_t < \Phi_t^{-1}(\alpha), \forall t\} = \alpha, \tag{11.33}$$

$$\mathcal{M}\{X_t \ge \Phi_t^{-1}(\alpha), \forall t\} = 1 - \alpha. \tag{11.34}$$

Please mention that $\{X_t < \Phi_t^{-1}(\alpha), \forall t\}$ and $\{X_t \ge \Phi_t^{-1}(\alpha), \forall t\}$ are disjoint events but not opposite. Although it is always true that

$$\mathcal{M}\{X_t < \Phi_t^{-1}(\alpha), \forall t\} + \mathcal{M}\{X_t \ge \Phi_t^{-1}(\alpha), \forall t\} \equiv 1, \tag{11.35}$$

the union of $\{X_t < \Phi_t^{-1}(\alpha), \forall t\}$ and $\{X_t \ge \Phi_t^{-1}(\alpha), \forall t\}$ does not make the universal set, and it is possible that

$$\mathcal{M}\{(X_t < \Phi_t^{-1}(\alpha),\ \forall t) \cup (X_t \ge \Phi_t^{-1}(\alpha),\ \forall t)\} < 1. \tag{11.36}$$

11.5 Extreme Value Theorem

This section will present a series of extreme value theorems for sample-continuous independent increment processes.

Theorem 11.10 *(Liu [89], Extreme Value Theorem) Let X_t be a sample-continuous independent increment process with uncertainty distribution $\Phi_t(x)$. Then the supremum*

$$\sup_{0 \le t \le s} X_t \tag{11.37}$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0 \le t \le s} \Phi_t(x); \tag{11.38}$$

and the infimum

$$\inf_{0 \le t \le s} X_t \tag{11.39}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0 \le t \le s} \Phi_t(x). \tag{11.40}$$

Proof: Let $0 = t_1 < t_2 < \cdots < t_n = s$ be a partition of the closed interval $[0, s]$. It is clear that

$$X_{t_i} = X_{t_1} + (X_{t_2} - X_{t_1}) + \cdots + (X_{t_i} - X_{t_{i-1}})$$

for $i = 1, 2, \cdots, n$. Since the increments

$$X_{t_1},\ X_{t_2} - X_{t_1},\ \cdots, X_{t_n} - X_{t_{n-1}}$$

are independent uncertain variables, it follows from Theorem 2.18 that the maximum

$$\max_{1\le i\le n} X_{t_i}$$

has an uncertainty distribution

$$\min_{1\le i\le n} \Phi_{t_i}(x).$$

Since X_t is sample-continuous, we have

$$\max_{1\le i\le n} X_{t_i} \to \sup_{0\le t\le s} X_t$$

and

$$\min_{1\le i\le n} \Phi_{t_i}(x) \to \inf_{0\le t\le s} \Phi_t(x)$$

as $n \to \infty$. Thus (11.38) is proved. Similarly, it follows from Theorem 2.18 that the minimum

$$\min_{1\le i\le n} X_{t_i}$$

has an uncertainty distribution

$$\max_{1\le i\le n} \Phi_{t_i}(x).$$

Since X_t is sample-continuous, we have

$$\min_{1\le i\le n} X_{t_i} \to \inf_{0\le t\le s} X_t$$

and

$$\max_{1\le i\le n} \Phi_{t_i}(x) \to \sup_{0\le t\le s} \Phi_t(x)$$

as $n \to \infty$. Thus (11.40) is verified.

Example 11.13: The sample-continuity condition in Theorem 11.10 cannot be removed. For example, take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Define a sample-discontinuous uncertain process

$$X_t(\gamma) = \begin{cases} 0, & \text{if } \gamma \ne t \\ 1, & \text{if } \gamma = t. \end{cases} \tag{11.41}$$

Since all increments are 0 almost surely, X_t is an independent increment process. On the one hand, X_t has an uncertainty distribution

$$\Phi_t(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \ge 0. \end{cases} \tag{11.42}$$

On the other hand, the supremum

$$\sup_{0 \le t \le 1} X_t(\gamma) \equiv 1 \tag{11.43}$$

has an uncertainty distribution

$$\Psi(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1, & \text{if } x \ge 1. \end{cases} \tag{11.44}$$

Thus

$$\Psi(x) \ne \inf_{0 \le t \le 1} \Phi_t(x). \tag{11.45}$$

Therefore, the sample-continuity condition cannot be removed.

Exercise 11.7: Let X_t be a sample-continuous independent increment process with uncertainty distribution $\Phi_t(x)$. Assume f is a continuous and strictly increasing function. Show that the supremum

$$\sup_{0 \le t \le s} f(X_t) \tag{11.46}$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0 \le t \le s} \Phi_t(f^{-1}(x)); \tag{11.47}$$

and the infimum

$$\inf_{0 \le t \le s} f(X_t) \tag{11.48}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0 \le t \le s} \Phi_t(f^{-1}(x)). \tag{11.49}$$

Exercise 11.8: Let X_t be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Assume f is a continuous and strictly decreasing function. Show that the supremum

$$\sup_{0 \le t \le s} f(X_t) \tag{11.50}$$

has an uncertainty distribution

$$\Psi(x) = 1 - \sup_{0 \le t \le s} \Phi_t(f^{-1}(x)); \tag{11.51}$$

and the infimum

$$\inf_{0\le t\le s} f(X_t) \tag{11.52}$$

has an uncertainty distribution

$$\Psi(x) = 1 - \inf_{0\le t\le s} \Phi_t(f^{-1}(x)). \tag{11.53}$$

11.6 First Hitting Time

Definition 11.9 *(Liu [89]) Let X_t be an uncertain process and let z be a given level. Then the uncertain variable*

$$\tau_z = \inf\left\{t \ge 0 \,\middle|\, X_t = z\right\} \tag{11.54}$$

is called the first hitting time that X_t reaches the level z.

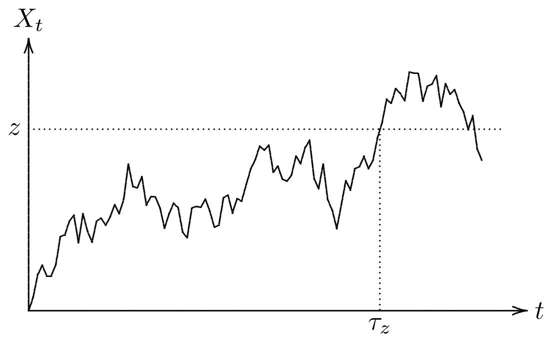


Figure 11.4: First Hitting Time

Theorem 11.11 *(Liu [89]) Let X_t be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Then the first hitting time τ_z that X_t reaches the level z has an uncertainty distribution,*

$$\Upsilon(s) = \begin{cases} 1 - \inf\limits_{0\le t\le s} \Phi_t(z), & \text{if } z > X_0 \\[2ex] \sup\limits_{0\le t\le s} \Phi_t(z), & \text{if } z < X_0. \end{cases} \tag{11.55}$$

Proof: When $X_0 < z$, it follows from the definition of first hitting time that

$$\tau_z \le s \text{ if and only if } \sup_{0\le t\le s} X_t \ge z.$$

Thus the uncertainty distribution of τ_z is

$$\Upsilon(s) = \mathcal{M}\{\tau_z \le s\} = \mathcal{M}\left\{\sup_{0\le t\le s} X_t \ge z\right\}.$$

By using the extreme value theorem, we obtain

$$\Upsilon(s) = 1 - \inf_{0 \le t \le s} \Phi_t(z).$$

When $X_0 > z$, it follows from the definition of first hitting time that

$$\tau_z \le s \text{ if and only if } \inf_{0 \le t \le s} X_t \le z.$$

Thus the uncertainty distribution of τ_z is

$$\Upsilon(s) = \mathcal{M}\{\tau_z \le s\} = \mathcal{M}\left\{\inf_{0 \le t \le s} X_t \le z\right\} = \sup_{0 \le t \le s} \Phi_t(z).$$

The theorem is verified.

Exercise 11.9: Let X_t be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Assume f is a continuous and strictly increasing function. Show that the first hitting time τ_z that $f(X_t)$ reaches the level z has an uncertainty distribution,

$$\Upsilon(s) = \begin{cases} 1 - \inf\limits_{0 \le t \le s} \Phi_t(f^{-1}(z)), & \text{if } z > f(X_0) \\ \sup\limits_{0 \le t \le s} \Phi_t(f^{-1}(z)), & \text{if } z < f(X_0). \end{cases} \tag{11.56}$$

Exercise 11.10: Let X_t be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Assume f is a continuous and strictly decreasing function. Show that the first hitting time τ_z that $f(X_t)$ reaches the level z has an uncertainty distribution,

$$\Upsilon(s) = \begin{cases} \sup\limits_{0 \le t \le s} \Phi_t(f^{-1}(z)), & \text{if } z > f(X_0) \\ 1 - \inf\limits_{0 \le t \le s} \Phi_t(f^{-1}(z)), & \text{if } z < f(X_0). \end{cases} \tag{11.57}$$

Exercise 11.11: Show that the sample-continuity condition in Theorem 11.11 cannot be removed.

11.7 Time Integral

This section will give a definition of time integral that is an integral of uncertain process with respect to time.

Definition 11.10 *(Liu [77]) Let X_t be an uncertain process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|. \tag{11.58}$$

Then the time integral of X_t with respect to t is

$$\int_a^b X_t \mathrm{d}t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (t_{i+1} - t_i) \tag{11.59}$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be time integrable.

Since X_t is an uncertain variable at each time t, the limit in (11.59) is also an uncertain variable provided that the limit exists almost surely and is finite. Hence an uncertain process X_t is time integrable if and only if the limit in (11.59) is an uncertain variable.

Theorem 11.12 *If X_t is a sample-continuous uncertain process on $[a, b]$, then it is time integrable on $[a, b]$.*

Proof: Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a partition of the closed interval $[a, b]$. Since the uncertain process X_t is sample-continuous, almost all sample paths are continuous functions with respect to t. Hence the limit

$$\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i)$$

exists almost surely and is finite. On the other hand, since X_t is an uncertain variable at each time t, the above limit is also a measurable function. Hence the limit is an uncertain variable and then X_t is time integrable.

Theorem 11.13 *If X_t is a time integrable uncertain process on $[a, b]$, then it is time integrable on each subinterval of $[a, b]$. Moreover, if $c \in [a, b]$, then*

$$\int_a^b X_t \mathrm{d}t = \int_a^c X_t \mathrm{d}t + \int_c^b X_t \mathrm{d}t. \tag{11.60}$$

Proof: Let $[a', b']$ be a subinterval of $[a, b]$. Since X_t is a time integrable uncertain process on $[a, b]$, for any partition

$$a = t_1 < \cdots < t_m = a' < t_{m+1} < \cdots < t_n = b' < t_{n+1} < \cdots < t_{k+1} = b,$$

the limit

$$\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i)$$

exists almost surely and is finite. Thus the limit

$$\lim_{\Delta \to 0} \sum_{i=m}^{n-1} X_{t_i}(t_{i+1} - t_i)$$

exists almost surely and is finite. Hence X_t is time integrable on the subinterval $[a', b']$. Next, for the partition

$$a = t_1 < \cdots < t_m = c < t_{m+1} < \cdots < t_{k+1} = b,$$

we have

$$\sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i) = \sum_{i=1}^{m-1} X_{t_i}(t_{i+1} - t_i) + \sum_{i=m}^{k} X_{t_i}(t_{i+1} - t_i).$$

Note that

$$\int_a^b X_t \mathrm{d}t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i),$$

$$\int_a^c X_t \mathrm{d}t = \lim_{\Delta \to 0} \sum_{i=1}^{m-1} X_{t_i}(t_{i+1} - t_i),$$

$$\int_c^b X_t \mathrm{d}t = \lim_{\Delta \to 0} \sum_{i=m}^{k} X_{t_i}(t_{i+1} - t_i).$$

Hence the equation (11.60) is proved.

Theorem 11.14 *(Linearity of Time Integral) Let X_t and Y_t be time integrable uncertain processes on $[a, b]$, and let α and β be real numbers. Then*

$$\int_a^b (\alpha X_t + \beta Y_t) \mathrm{d}t = \alpha \int_a^b X_t \mathrm{d}t + \beta \int_a^b Y_t \mathrm{d}t. \tag{11.61}$$

Proof: Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a partition of the closed interval $[a, b]$. It follows from the definition of time integral that

$$\begin{aligned}
&\int_a^b (\alpha X_t + \beta Y_t) \mathrm{d}t = \lim_{\Delta \to 0} \sum_{i=1}^{k} (\alpha X_{t_i} + \beta Y_{t_i})(t_{i+1} - t_i) \\
&= \lim_{\Delta \to 0} \alpha \sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i) + \lim_{\Delta \to 0} \beta \sum_{i=1}^{k} Y_{t_i}(t_{i+1} - t_i) \\
&= \alpha \int_a^b X_t \mathrm{d}t + \beta \int_a^b Y_t \mathrm{d}t.
\end{aligned}$$

Hence the equation (11.61) is proved.

Theorem 11.15 *(Yao [186]) Let X_t be a sample-continuous independent increment process with regular uncertainty distribution $\Phi_t(x)$. Then the time integral*

$$Y_s = \int_0^s X_t \mathrm{d}t \tag{11.62}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s \Phi_t^{-1}(\alpha)\mathrm{d}t. \tag{11.63}$$

Proof: For any given time $s > 0$, it follows from the basic property of time integral that

$$\left\{\int_0^s X_t\mathrm{d}t \le \int_0^s \Phi_t^{-1}(\alpha)\mathrm{d}t\right\} \supset \{X_t \le \Phi_t^{-1}(\alpha), \forall t\}.$$

By using Theorem 11.9, we obtain

$$\mathcal{M}\left\{\int_0^s X_t\mathrm{d}t \le \int_0^s \Phi_t^{-1}(\alpha)\mathrm{d}t\right\} \ge \mathcal{M}\{X_t \le \Phi_t^{-1}(\alpha), \forall t\} = \alpha.$$

Similarly, since

$$\left\{\int_0^s X_t\mathrm{d}t > \int_0^s \Phi_t^{-1}(\alpha)\mathrm{d}t\right\} \supset \{X_t > \Phi_t^{-1}(\alpha), \forall t\},$$

we have

$$\mathcal{M}\left\{\int_0^s X_t\mathrm{d}t > \int_0^s \Phi_t^{-1}(\alpha)\mathrm{d}t\right\} \ge \mathcal{M}\{X_t > \Phi_t^{-1}(\alpha), \forall t\} = 1-\alpha.$$

It follows from the above two inequalities and the duality axiom that

$$\mathcal{M}\left\{\int_0^s X_t\mathrm{d}t \le \int_0^s \Phi_t^{-1}(\alpha)\mathrm{d}t\right\} = \alpha.$$

Thus the time integral Y_s has the inverse uncertainty distribution $\Psi_s^{-1}(\alpha)$.

Exercise 11.12: Let X_t be a sample-continuous independent increment process with regular uncertainty distribution $\Phi_t(x)$, and let $J(x)$ be a strictly increasing function. Show that the time integral

$$Y_s = \int_0^s J(X_t)\mathrm{d}t \tag{11.64}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s J(\Phi_t^{-1}(\alpha))\mathrm{d}t. \tag{11.65}$$

Exercise 11.13: Let X_t be a sample-continuous independent increment process with regular uncertainty distribution $\Phi_t(x)$, and let $J(x)$ be a strictly decreasing function. Show that the time integral

$$Y_s = \int_0^s J(X_t)\mathrm{d}t \tag{11.66}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s J(\Phi_t^{-1}(1-\alpha))\mathrm{d}t. \tag{11.67}$$

11.8 Stationary Increment Process

An uncertain process X_t is said to have *stationary increments* if its increments are identically distributed uncertain variables whenever the time intervals have the same length, i.e., for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.

Definition 11.11 *(Liu [77]) An uncertain process is said to be a stationary independent increment process if it has not only stationary increments but also independent increments.*

It is clear that a stationary independent increment process is a special independent increment process.

Theorem 11.16 *Let X_t be a stationary independent increment process. Then for any real numbers a and b, the uncertain process*

$$Y_t = aX_t + b \tag{11.68}$$

is also a stationary independent increment process.

Proof: Since X_t is an independent increment process, the uncertain variables

$$X_{t_1},\ X_{t_2} - X_{t_1},\ X_{t_3} - X_{t_2},\ \cdots,\ X_{t_k} - X_{t_{k-1}}$$

are independent. It follows from $Y_t = aX_t + b$ and Theorem 2.7 that

$$Y_{t_1},\ Y_{t_2} - Y_{t_1},\ Y_{t_3} - Y_{t_2},\ \cdots,\ Y_{t_k} - Y_{t_{k-1}}$$

are also independent. That is, Y_t is an independent increment process. On the other hand, since X_t is a stationary increment process, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$. Thus

$$Y_{s+t} - Y_s = a(X_{s+t} - X_s)$$

are also identically distributed uncertain variables for all $s > 0$, and Y_t is a stationary increment process. Hence Y_t is a stationary independent increment process.

Remark 11.4: Generally speaking, a nonlinear function of stationary independent increment process is not necessarily a stationary independent increment process. A typical example is the square of a stationary independent increment process.

Theorem 11.17 *(Chen [10]) Suppose X_t is a stationary independent increment process. Then X_t and $(1-t)X_0 + tX_1$ are identically distributed uncertain variables for any time $t \geq 0$.*

Proof: We first prove the theorem when t is a rational number. Assume $t = q/p$ where p and q are irreducible integers. Let Φ be the common uncertainty distribution of increments

$$X_{1/p} - X_{0/p},\ X_{2/p} - X_{1/p},\ X_{3/p} - X_{2/p},\ \cdots$$

Then

$$X_t - X_0 = (X_{1/p} - X_{0/p}) + (X_{2/p} - X_{1/p}) + \cdots + (X_{q/p} - X_{(q-1)/p})$$

has an uncertainty distribution

$$\Psi(x) = \Phi(x/q). \tag{11.69}$$

In addition,

$$t(X_1 - X_0) = t((X_{1/p} - X_{0/p}) + (X_{2/p} - X_{1/p}) + \cdots + (X_{p/p} - X_{(p-1)/p}))$$

has an uncertainty distribution

$$\Upsilon(x) = \Phi(x/p/t) = \Phi(x/p/(q/p)) = \Phi(x/q). \tag{11.70}$$

It follows from (11.69) and (11.70) that $X_t - X_0$ and $t(X_1 - X_0)$ are identically distributed, and so are X_t and $(1-t)X_0 + tX_1$.

Remark 11.5: If X_t is a stationary independent increment process with $X_0 = 0$, then X_t/t and X_1 are identically distributed uncertain variables. In other words, there is an uncertainty distribution Φ such that

$$\frac{X_t}{t} \sim \Phi(x) \tag{11.71}$$

or equivalently,

$$X_t \sim \Phi\left(\frac{x}{t}\right) \tag{11.72}$$

for any time $t > 0$. Note that Φ is just the uncertainty distribution of X_1.

Theorem 11.18 *(Liu [93]) Let X_t be a stationary independent increment process whose initial value and increments have inverse uncertainty distributions. Then there exist two continuous and strictly increasing functions μ and ν such that X_t has an inverse uncertainty distribution*

$$\Phi_t^{-1}(\alpha) = \mu(\alpha) + \nu(\alpha)t. \tag{11.73}$$

Proof: Note that X_0 and $X_1 - X_0$ are independent uncertain variables whose inverse uncertainty distributions exist and are denoted by $\mu(\alpha)$ and $\nu(\alpha)$, respectively. It is clear that $\mu(\alpha)$ and $\nu(\alpha)$ are continuous and strictly increasing functions. Furthermore, it follows from Theorem 11.17 that X_t and $X_0 + (X_1 - X_0)t$ are identically distributed uncertain variables. Hence

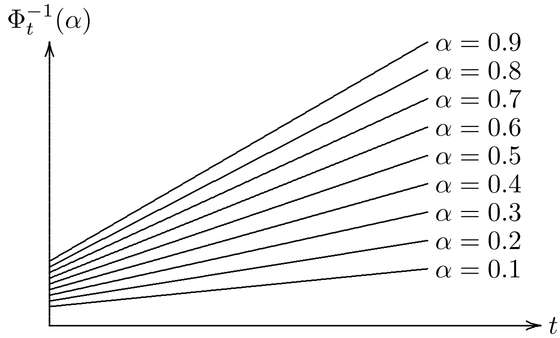


Figure 11.5: Inverse Uncertainty Distribution of Stationary Independent Increment Process

X_t has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha) = \mu(\alpha) + \nu(\alpha)t$. The theorem is verified.

Remark 11.6: The inverse uncertainty distribution of stationary independent increment process is a family of linear functions of t indexed by α. See Figure 11.5.

Theorem 11.19 *(Liu [93]) Let μ and ν be continuous and strictly increasing functions on $(0,1)$. Then there exists a stationary independent increment process X_t whose inverse uncertainty distribution is*

$$\Phi_t^{-1}(\alpha) = \mu(\alpha) + \nu(\alpha)t. \tag{11.74}$$

Furthermore, X_t has a Lipschitz continuous version.

Proof: Without loss of generality, we only consider the range of $t \in [0,1]$. Let

$$\left\{\xi(r) \;\middle|\; r \text{ represents rational numbers in } [0,1]\right\}$$

be a countable sequence of independent uncertain variables, where $\xi(0)$ has an inverse uncertainty distribution $\mu(\alpha)$ and $\xi(r)$ have a common inverse uncertainty distribution $\nu(\alpha)$ for all rational numbers r in $(0,1]$. For each positive integer n, we define an uncertain process

$$X_t^n = \begin{cases} \xi(0) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{k} \xi\left(\dfrac{i}{n}\right), & \text{if } t = \dfrac{k}{n} \quad (k = 1, 2, \cdots, n) \\ \text{linear}, & \text{otherwise.} \end{cases}$$

It may prove that X_t^n converges in distribution as $n \to \infty$. Furthermore, we may verify that the limit is a stationary independent increment process and has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. The theorem is verified.

Theorem 11.20 *(Liu [83]) Let X_t be a stationary independent increment process. Then there exist two real numbers a and b such that*

$$E[X_t] = a + bt \tag{11.75}$$

for any time $t \geq 0$.

Proof: It follows from Theorem 11.17 that X_t and $X_0 + (X_1 - X_0)t$ are identically distributed uncertain variables. Thus we have

$$E[X_t] = E[X_0 + (X_1 - X_0)t].$$

Since X_0 and $X_1 - X_0$ are independent uncertain variables, we obtain

$$E[X_t] = E[X_0] + E[X_1 - X_0]t.$$

Hence (11.75) holds for $a = E[X_0]$ and $b = E[X_1 - X_0]$.

Theorem 11.21 *(Liu [83]) Let X_t be a stationary independent increment process with an initial value 0. Then for any times s and t, we have*

$$E[X_{s+t}] = E[X_s] + E[X_t]. \tag{11.76}$$

Proof: It follows from Theorem 11.20 that there exists a real number b such that $E[X_t] = bt$ for any time $t \geq 0$. Hence

$$E[X_{s+t}] = b(s + t) = bs + bt = E[X_s] + E[X_t].$$

Theorem 11.22 *(Chen [10]) Let X_t be a stationary independent increment process with a crisp initial value X_0. Then there exists a real number b such that*

$$V[X_t] = bt^2 \tag{11.77}$$

for any time $t \geq 0$.

Proof: It follows from Theorem 11.17 that X_t and $(1 - t)X_0 + tX_1$ are identically distributed uncertain variables. Since X_0 is a constant, we have

$$V[X_t] = V[(1 - t)X_0 + tX_1] = t^2 V[X_1].$$

Hence (11.77) holds for $b = V[X_1]$.

Theorem 11.23 *(Chen [10]) Let X_t be a stationary independent increment process with a crisp initial value X_0. Then for any times s and t, we have*

$$\sqrt{V[X_{s+t}]} = \sqrt{V[X_s]} + \sqrt{V[X_t]}. \tag{11.78}$$

Proof: It follows from Theorem 11.22 that there exists a real number b such that $V[X_t] = bt^2$ for any time $t \geq 0$. Hence

$$\sqrt{V[X_{s+t}]} = \sqrt{b}(s + t) = \sqrt{b}s + \sqrt{b}t = \sqrt{V[X_s]} + \sqrt{V[X_t]}.$$

11.9 Bibliographic Notes

The study of uncertain process was started by Liu [77] in 2008 for modelling the evolution of uncertain phenomena. In order to describe uncertain process, Liu [93] proposed the uncertainty distribution and inverse uncertainty distribution. In addition, the independence concept of uncertain processes was introduced by Liu [93].

Independent increment process was initialized by Liu [77], and a sufficient and necessary condition was proved by Liu [93] for its inverse uncertainty distribution. In addition, Liu [89] presented an extreme value theorem and obtained the uncertainty distribution of first hitting time, and Yao [186] provided a formula for calculating the inverse uncertainty distribution of time integral of independent increment process.

Stationary independent increment process was initialized by Liu [77], and its inverse uncertainty distribution was investigated by Liu [93]. Furthermore, Liu [83] showed that the expected value is a linear function of time, and Chen [10] verified that the variance is proportional to the square of time.

Chapter 12

Uncertain Renewal Process

Uncertain renewal process is an uncertain process in which events occur continuously and independently of one another in uncertain times. This chapter will introduce uncertain renewal process, renewal reward process, and alternating renewal process. This chapter will also provide block replacement policy, age replacement policy, and an uncertain insurance model.

12.1 Uncertain Renewal Process

Definition 12.1 *(Liu [77]) Let $\xi_1, \xi_2, \cdots$ be iid uncertain interarrival times. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then the uncertain process*

$$N_t = \max_{n \geq 0} \{n \mid S_n \leq t\} \tag{12.1}$$

is called an uncertain renewal process.

It is clear that S_n is a stationary independent increment process with respect to n. Since $\xi_1, \xi_2, \cdots$ denote the interarrival times of successive events, S_n can be regarded as the waiting time until the occurrence of the nth event. In this case, the renewal process N_t is the number of renewals in $(0, t]$. Note that N_t is not sample-continuous, but each sample path of N_t is a right-continuous and increasing step function taking only nonnegative integer values. Furthermore, since the interarrival times are always assumed to be positive uncertain variables, the size of each jump of N_t is always 1. In other words, N_t has at most one renewal at each time. In particular, N_t does not jump at time 0.

Theorem 12.1 *(Fundamental Relationship) Let N_t be a renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$, and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$.*

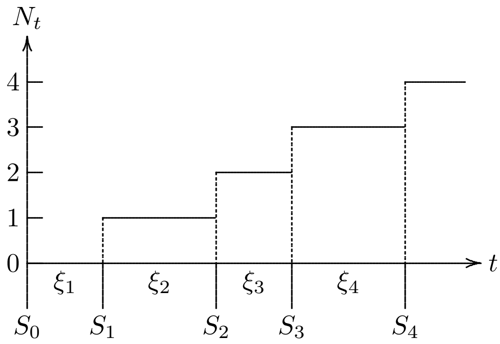


Figure 12.1: A Sample Path of Renewal Process

Then we have

$$N_t \ge n \Leftrightarrow S_n \le t \tag{12.2}$$

for any time t and integer n. Furthermore, we also have

$$N_t \le n \Leftrightarrow S_{n+1} > t. \tag{12.3}$$

Proof: Since N_t is the largest n such that $S_n \le t$, we have $S_{N_t} \le t < S_{N_t+1}$. If $N_t \ge n$, then $S_n \le S_{N_t} \le t$. Conversely, if $S_n \le t$, then $S_n < S_{N_t+1}$ that implies $N_t \ge n$. Thus (12.2) is verified. Similarly, if $N_t \le n$, then $N_t + 1 \le n + 1$ and $S_{n+1} \ge S_{N_t+1} > t$. Conversely, if $S_{n+1} > t$, then $S_{n+1} > S_{N_t}$ that implies $N_t \le n$. Thus (12.3) is verified.

Exercise 12.1: Let N_t be a renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$, and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$. Show that

$$\mathcal{M}\{N_t \ge n\} = \mathcal{M}\{S_n \le t\}, \tag{12.4}$$

$$\mathcal{M}\{N_t \le n\} = 1 - \mathcal{M}\{S_{n+1} \le t\}. \tag{12.5}$$

Theorem 12.2 *(Liu [83]) Let N_t be a renewal process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ If Φ is the common uncertainty distribution of those interarrival times, then N_t has an uncertainty distribution*

$$\Upsilon_t(x) = 1 - \Phi\left(\frac{t}{\lfloor x \rfloor + 1}\right), \quad \forall x \ge 0 \tag{12.6}$$

where $\lfloor x \rfloor$ represents the maximal integer less than or equal to x.

Proof: Note that S_{n+1} has an uncertainty distribution $\Phi(x/(n+1))$. It follows from (12.5) that

$$\mathcal{M}\{N_t \le n\} = 1 - \mathcal{M}\{S_{n+1} \le t\} = 1 - \Phi\left(\frac{t}{n+1}\right).$$

Since N_t takes integer values, for any $x \geq 0$, we have

$$\Upsilon_t(x) = \mathcal{M}\{N_t \leq x\} = \mathcal{M}\{N_t \leq \lfloor x \rfloor\} = 1 - \Phi\left(\frac{t}{\lfloor x \rfloor + 1}\right).$$

The theorem is verified.

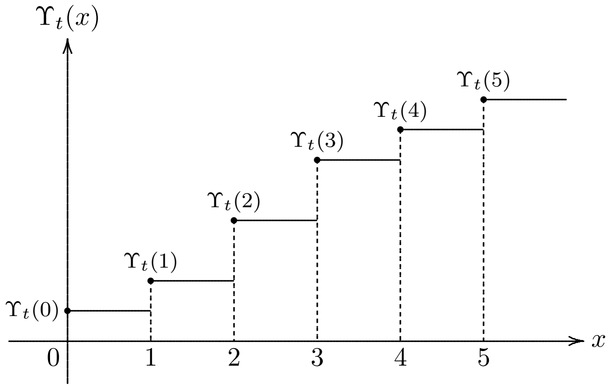


Figure 12.2: Uncertainty Distribution $\Upsilon_t(x)$ of Renewal Process N_t

Theorem 12.3 *(Liu [83], Elementary Renewal Theorem) Let N_t be a renewal process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ Then the average renewal number*

$$\frac{N_t}{t} \to \frac{1}{\xi_1} \tag{12.7}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: The uncertainty distribution Υ_t of N_t has been given by Theorem 12.2 as follows,

$$\Upsilon_t(x) = 1 - \Phi\left(\frac{t}{\lfloor x \rfloor + 1}\right)$$

where Φ is the uncertainty distribution of ξ_1. It follows from the operational law that the uncertainty distribution of N_t/t is

$$\Psi_t(x) = 1 - \Phi\left(\frac{t}{\lfloor tx \rfloor + 1}\right)$$

where $\lfloor tx \rfloor$ represents the maximal integer less than or equal to tx. Thus at each continuity point x of $1 - \Phi(1/x)$, we have

$$\lim_{t \to \infty} \Psi_t(x) = 1 - \Phi\left(\frac{1}{x}\right)$$

which is just the uncertainty distribution of $1/\xi_1$. Hence N_t/t converges in distribution to $1/\xi_1$ as $t \to \infty$.

Theorem 12.4 *(Liu [83], Elementary Renewal Theorem) Let N_t be a renewal process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ Then*

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = E\left[\frac{1}{\xi_1}\right]. \tag{12.8}$$

If Φ is the common uncertainty distribution of those interarrival times, then

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = \int_0^{+\infty} \Phi\left(\frac{1}{x}\right) \mathrm{d}x. \tag{12.9}$$

If the uncertainty distribution Φ is regular, then

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = \int_0^1 \frac{1}{\Phi^{-1}(\alpha)} \mathrm{d}\alpha. \tag{12.10}$$

Proof: Write the uncertainty distributions of N_t/t and $1/\xi_1$ by $\Psi_t(x)$ and $G(x)$, respectively. Theorem 12.3 says that $\Psi_t(x) \to G(x)$ as $t \to \infty$ at each continuity point x of $G(x)$. Note that $\Psi_t(x) \ge G(x)$. It follows from the Lebesgue dominated convergence theorem and the existence of $E[1/\xi_1]$ that

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = \lim_{t\to\infty} \int_0^{+\infty} (1 - \Psi_t(x))\mathrm{d}x = \int_0^{+\infty} (1 - G(x))\mathrm{d}x = E\left[\frac{1}{\xi_1}\right].$$

Since $1/\xi_1$ has an uncertainty distribution $1 - \Phi(1/x)$, we have

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = E\left[\frac{1}{\xi_1}\right] = \int_0^{+\infty} \Phi\left(\frac{1}{x}\right) \mathrm{d}x.$$

Furthermore, since $1/\xi_1$ has an inverse uncertainty distribution

$$G^{-1}(\alpha) = \frac{1}{\Phi^{-1}(1-\alpha)},$$

we get

$$E\left[\frac{1}{\xi_1}\right] = \int_0^1 \frac{1}{\Phi^{-1}(1-\alpha)} \mathrm{d}\alpha = \int_0^1 \frac{1}{\Phi^{-1}(\alpha)} \mathrm{d}\alpha.$$

The theorem is proved.

Exercise 12.2: A renewal process N_t is called *linear* if $\xi_1, \xi_2, \cdots$ are iid linear uncertain variables $\mathcal{L}(a, b)$ with $a > 0$. Show that

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = \frac{\ln b - \ln a}{b - a}. \tag{12.11}$$

Exercise 12.3: A renewal process N_t is called *zigzag* if $\xi_1, \xi_2, \cdots$ are iid zigzag uncertain variables $\mathcal{Z}(a, b, c)$ with $a > 0$. Show that

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = \frac{1}{2}\left(\frac{\ln b - \ln a}{b - a} + \frac{\ln c - \ln b}{c - b}\right). \tag{12.12}$$

Exercise 12.4: A renewal process N_t is called *lognormal* if $\xi_1, \xi_2, \cdots$ are iid lognormal uncertain variables $\mathcal{LOGN}(e, \sigma)$. Show that

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = \begin{cases} \sqrt{3}\sigma \exp(-e) \csc(\sqrt{3}\sigma), & \text{if } \sigma < \pi/\sqrt{3} \\ +\infty, & \text{if } \sigma \geq \pi/\sqrt{3}. \end{cases} \tag{12.13}$$

12.2 Block Replacement Policy

Block replacement policy means that an element is always replaced at failure or periodically with time s. Assume that the lifetimes of elements are iid uncertain variables $\xi_1, \xi_2, \cdots$ with a common uncertainty distribution Φ. Then the replacement times form an uncertain renewal process N_t. Let a denote the "failure replacement" cost of replacing an element when it fails earlier than s, and b the "planned replacement" cost of replacing an element at planned time s. Note that $a > b > 0$ is always assumed. It is clear that the cost of one period is $aN_s + b$ and the average cost is

$$\frac{aN_s + b}{s}. \tag{12.14}$$

Theorem 12.5 *(Ke-Yao [66]) Assume the lifetimes of elements are iid uncertain variables $\xi_1, \xi_2, \cdots$ with a common uncertainty distribution Φ, and N_t is the uncertain renewal process representing the replacement times. Then the average cost has an expected value*

$$E\left[\frac{aN_s + b}{s}\right] = \frac{1}{s}\left(a\sum_{n=1}^{\infty} \Phi\left(\frac{s}{n}\right) + b\right). \tag{12.15}$$

Proof: Note that the uncertainty distribution of N_t is a step function. It follows from Theorem 12.2 that

$$E[N_s] = \int_0^{+\infty} \Phi\left(\frac{s}{\lfloor x\rfloor + 1}\right) \mathrm{d}x = \sum_{n=1}^{\infty} \Phi\left(\frac{s}{n}\right).$$

Thus (12.15) is verified by

$$E\left[\frac{aN_s + b}{s}\right] = \frac{aE[N_s] + b}{s}. \tag{12.16}$$

What is the optimal time s?

When the block replacement policy is accepted, one problem is concerned with finding an optimal time s in order to minimize the average cost, i.e.,

$$\min_s \frac{1}{s}\left(a\sum_{n=1}^{\infty} \Phi\left(\frac{s}{n}\right) + b\right). \tag{12.17}$$

12.3 Renewal Reward Process

Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \cdots$ be a sequence of pairs of uncertain variables. We shall interpret η_i as the rewards (or costs) associated with the i-th interarrival times ξ_i for $i = 1, 2, \cdots$, respectively.

Definition 12.2 *(Liu [83]) Let $\xi_1, \xi_2, \cdots$ be iid uncertain interarrival times, and let $\eta_1, \eta_2, \cdots$ be iid uncertain rewards. Then*

$$R_t = \sum_{i=1}^{N_t} \eta_i \tag{12.18}$$

is called a renewal reward process, where N_t is the renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$

A renewal reward process R_t denotes the total reward earned by time t. In addition, if $\eta_i \equiv 1$, then R_t degenerates to a renewal process N_t. Please also note that $R_t = 0$ whenever $N_t = 0$.

Theorem 12.6 *(Liu [83]) Let R_t be a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid uncertain rewards $\eta_1, \eta_2, \cdots$ Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors, and those interarrival times and rewards have uncertainty distributions Φ and Ψ, respectively. Then R_t has an uncertainty distribution*

$$\Upsilon_t(x) = \max_{k \ge 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{x}{k}\right). \tag{12.19}$$

Here we set $x/k = +\infty$ and $\Psi(x/k) = 1$ when $k = 0$.

Proof: It follows from the definition of renewal reward process that the renewal process N_t is independent of uncertain rewards $\eta_1, \eta_2, \cdots$, and R_t has an uncertainty distribution

$$\begin{aligned}
\Upsilon_t(x) &= \mathcal{M}\left\{\sum_{i=1}^{N_t} \eta_i \le x\right\} = \mathcal{M}\left\{\bigcup_{k=0}^{\infty} (N_t = k) \cap \sum_{i=1}^{k} \eta_i \le x\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty} (N_t \le k) \cap \sum_{i=1}^{k} \eta_i \le x\right\} \quad \text{(this is a polyrectangle)} \\
&= \max_{k \ge 0} \mathcal{M}\left\{(N_t \le k) \cap \sum_{i=1}^{k} \eta_i \le x\right\} \quad \text{(polyrectangular theorem)} \\
&= \max_{k \ge 0} \mathcal{M}\{N_t \le k\} \wedge \mathcal{M}\left\{\sum_{i=1}^{k} \eta_i \le x\right\} \quad \text{(independence)} \\
&= \max_{k \ge 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{x}{k}\right).
\end{aligned}$$

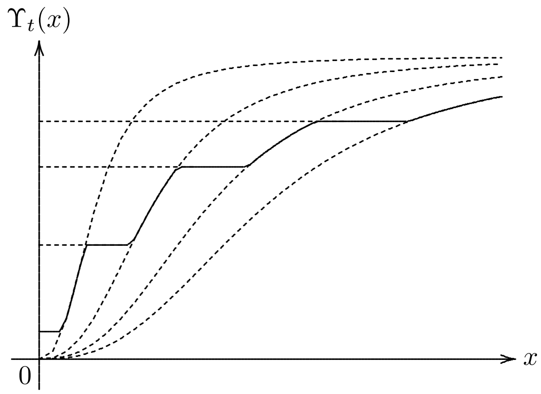


Figure 12.3: Uncertainty Distribution $\Upsilon_t(x)$ of Renewal Reward Process R_t in which the dashed horizontal lines are $1 - \Phi(t/(k+1))$ and the dashed curves are $\Psi(x/k)$ for $k = 0, 1, 2, \cdots$

The theorem is proved.

Theorem 12.7 *(Liu [83], Renewal Reward Theorem) Let R_t be a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid uncertain rewards $\eta_1, \eta_2, \cdots$ Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors. Then the reward rate*

$$\frac{R_t}{t} \to \frac{\eta_1}{\xi_1} \tag{12.20}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: Assume those interarrival times and rewards have uncertainty distributions Φ and Ψ, respectively. It follows from Theorem 12.6 that the uncertainty distribution of R_t is

$$\Upsilon_t(x) = \max_{k \geq 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{x}{k}\right).$$

Then R_t/t has an uncertainty distribution

$$\Psi_t(x) = \max_{k \geq 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{tx}{k}\right).$$

When $t \to \infty$, we have

$$\Psi_t(x) \to \sup_{y \geq 0} (1 - \Phi(y)) \wedge \Psi(xy)$$

which is just the uncertainty distribution of η_1/ξ_1. Hence R_t/t converges in distribution to η_1/ξ_1 as $t \to \infty$.

Theorem 12.8 *(Liu [83], Renewal Reward Theorem) Let R_t be a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid uncertain rewards $\eta_1, \eta_2, \cdots$ Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors. Then*

$$\lim_{t\to\infty} \frac{E[R_t]}{t} = E\left[\frac{\eta_1}{\xi_1}\right]. \tag{12.21}$$

If those interarrival times and rewards have regular uncertainty distributions Φ and Ψ, respectively, then

$$\lim_{t\to\infty} \frac{E[R_t]}{t} = \int_0^1 \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)} \mathrm{d}\alpha. \tag{12.22}$$

Proof: It follows from Theorem 12.6 that R_t/t has an uncertainty distribution

$$F_t(x) = \max_{k\ge 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{tx}{k}\right)$$

and η_1/ξ_1 has an uncertainty distribution

$$G(x) = \sup_{y\ge 0} (1-\Phi(y)) \wedge \Psi(xy).$$

Note that $F_t(x) \to G(x)$ and $F_t(x) \ge G(x)$. It follows from Lebesgue dominated convergence theorem and the existence of $E[\eta_1/\xi_1]$ that

$$\lim_{t\to\infty} \frac{E[R_t]}{t} = \lim_{t\to\infty} \int_0^{+\infty} (1-F_t(x))\mathrm{d}x = \int_0^{+\infty} (1-G(x))\mathrm{d}x = E\left[\frac{\eta_1}{\xi_1}\right].$$

Finally, since η_1/ξ_1 has an inverse uncertainty distribution

$$G^{-1}(\alpha) = \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)},$$

we get

$$E\left[\frac{\eta_1}{\xi_1}\right] = \int_0^1 \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)} \mathrm{d}\alpha.$$

The theorem is proved.

12.4 Uncertain Insurance Model

Liu [89] assumed that a is the initial capital of an insurance company, b is the premium rate, bt is the total income up to time t, and the uncertain claim process is a renewal reward process

$$R_t = \sum_{i=1}^{N_t} \eta_i \tag{12.23}$$

with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid uncertain claim amounts $\eta_1, \eta_2, \cdots$ Then the capital of the insurance company at time t is

$$Z_t = a + bt - R_t \tag{12.24}$$

and Z_t is called an *insurance risk process.*

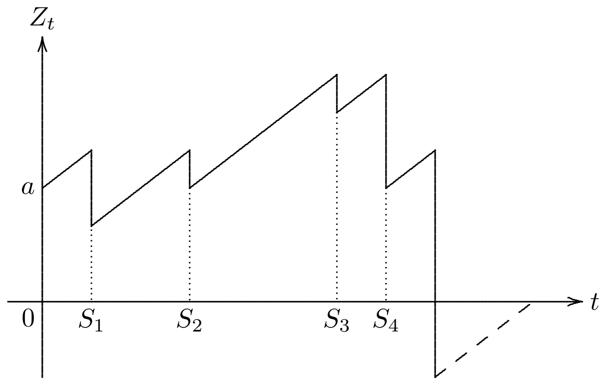


Figure 12.4: An Insurance Risk Process

Ruin Index

Ruin index is the uncertain measure that the capital of the insurance company becomes negative.

Definition 12.3 *(Liu [89]) Let Z_t be an insurance risk process. Then the ruin index is defined as the uncertain measure that Z_t eventually becomes negative, i.e.,*

$$Ruin = \mathcal{M}\left\{\inf_{t\geq 0} Z_t < 0\right\}. \tag{12.25}$$

It is clear that the ruin index is a special case of the risk index in the sense of Liu [82].

Theorem 12.9 *(Liu [89], Ruin Index Theorem) Let $Z_t = a + bt - R_t$ be an insurance risk process where a and b are positive numbers, and R_t is a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid uncertain claim amounts $\eta_1, \eta_2, \cdots$ Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors, and those interarrival times and claim amounts have continuous uncertainty distributions Φ and Ψ, respectively. Then the ruin index is*

$$Ruin = \max_{k\geq 1}\sup_{x\geq 0} \Phi\left(\frac{x-a}{kb}\right) \wedge \left(1 - \Psi\left(\frac{x}{k}\right)\right). \tag{12.26}$$

Proof: For each positive integer k, it is clear that the arrival time of the kth claim is

$$S_k = \xi_1 + \xi_2 + \cdots + \xi_k$$

whose uncertainty distribution is $\Phi(s/k)$. Define an uncertain process indexed by k as follows,

$$Y_k = a + bS_k - (\eta_1 + \eta_2 + \cdots + \eta_k).$$

It is easy to verify that Y_k is an independent increment process with respect to k. In addition, Y_k is just the capital at the arrival time S_k and has an uncertainty distribution

$$F_k(z) = \sup_{x\ge 0} \Phi\left(\frac{z+x-a}{kb}\right) \wedge \left(1-\Psi\left(\frac{x}{k}\right)\right).$$

Since a ruin occurs only at the arrival times, we have

$$Ruin = \mathcal{M}\left\{\inf_{t\ge 0} Z_t < 0\right\} = \mathcal{M}\left\{\min_{k\ge 1} Y_k < 0\right\}.$$

It follows from the extreme value theorem that

$$Ruin = \max_{k\ge 1} F_k(0) = \max_{k\ge 1}\sup_{x\ge 0} \Phi\left(\frac{x-a}{kb}\right) \wedge \left(1-\Psi\left(\frac{x}{k}\right)\right).$$

The theorem is proved.

Ruin Time

Definition 12.4 *(Liu [89]) Let Z_t be an insurance risk process. Then the ruin time is defined as the first hitting time that the total capital Z_t becomes negative, i.e.,*

$$\tau = \inf\left\{t \ge 0 \,\middle|\, Z_t < 0\right\}. \tag{12.27}$$

Theorem 12.10 *(Yao [182]) Let $Z_t = a + bt - R_t$ be an insurance risk process where a and b are positive numbers, and R_t is a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid uncertain claim amounts $\eta_1, \eta_2, \cdots$ Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors, and those interarrival times and claim amounts have continuous uncertainty distributions Φ and Ψ, respectively. Then the ruin time has an uncertainty distribution*

$$\Upsilon(t) = \max_{k\ge 1}\sup_{x\le t} \Phi\left(\frac{x}{k}\right) \wedge \left(1-\Psi\left(\frac{a+bx}{k}\right)\right). \tag{12.28}$$

Proof: For each positive integer k, let us write $S_k = \xi_1 + \xi_2 + \cdots + \xi_k$, $Y_k = a + bS_k - (\eta_1 + \eta_2 + \cdots + \eta_k)$ and

$$\alpha_k = \sup_{x \le t} \Phi\left(\frac{x}{k}\right) \wedge \left(1 - \Psi\left(\frac{a+bx}{k}\right)\right).$$

Then

$$\alpha_k = \sup\left\{\alpha \,|\, k\Phi^{-1}(\alpha) \le t\right\} \wedge \sup\left\{\alpha \,|\, a + k\Phi^{-1}(\alpha) - k\Psi^{-1}(1-\alpha) < 0\right\}.$$

On the one hand, it follows from the definition of the ruin time τ that for each t, we have

$$\tau \le t \text{ if and only if } \inf_{0 \le s \le t} Z_s < 0.$$

Thus

$$\begin{aligned}
\mathcal{M}\{\tau \le t\} &= \mathcal{M}\left\{\inf_{0\le s\le t} Z_s < 0\right\} = \mathcal{M}\left\{\bigcup_{k=1}^{\infty}(S_k \le t,\, Y_k < 0)\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=1}^{\infty}\left(\sum_{i=1}^{k}\xi_i \le t,\, a + b\sum_{i=1}^{k}\xi_i - \sum_{i=1}^{k}\eta_i < 0\right)\right\} \\
&\ge \mathcal{M}\left\{\bigcup_{k=1}^{\infty}\bigcap_{i=1}^{k}(\xi_i \le \Phi^{-1}(\alpha_k)) \cap (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&\ge \bigvee_{k=1}^{\infty}\mathcal{M}\left\{\bigcap_{i=1}^{k}(\xi_i \le \Phi^{-1}(\alpha_k)) \cap (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&= \bigvee_{k=1}^{\infty}\bigwedge_{i=1}^{k}\mathcal{M}\left\{(\xi_i \le \Phi^{-1}(\alpha_k)) \cap (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&= \bigvee_{k=1}^{\infty}\bigwedge_{i=1}^{k}\mathcal{M}\left\{\xi_i \le \Phi^{-1}(\alpha_k)\right\} \wedge \mathcal{M}\left\{\eta_i > \Psi^{-1}(1-\alpha_k)\right\} \\
&= \bigvee_{k=1}^{\infty}\bigwedge_{i=1}^{k}\alpha_k \wedge \alpha_k = \bigvee_{k=1}^{\infty}\alpha_k.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{M}\{\tau \le t\} &= \mathcal{M}\left\{\bigcup_{k=1}^{\infty}\left(\sum_{i=1}^{k}\xi_i \le t,\ a+b\sum_{i=1}^{k}\xi_i - \sum_{i=1}^{k}\eta_i < 0\right)\right\} \\
&\le \mathcal{M}\left\{\bigcup_{k=1}^{\infty}\bigcup_{i=1}^{k}(\xi_i \le \Phi^{-1}(\alpha_k)) \cup (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&= \mathcal{M}\left\{\bigcup_{i=1}^{\infty}\bigcup_{k=i}^{\infty}(\xi_i \le \Phi^{-1}(\alpha_k)) \cup (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&\le \mathcal{M}\left\{\bigcup_{i=1}^{\infty}\left(\xi_i \le \bigvee_{k=i}^{\infty}\Phi^{-1}(\alpha_k)\right) \cup \left(\eta_i > \bigwedge_{k=i}^{\infty}\Psi^{-1}(1-\alpha_k)\right)\right\} \\
&= \bigvee_{i=1}^{\infty}\mathcal{M}\left\{\xi_i \le \bigvee_{k=i}^{\infty}\Phi^{-1}(\alpha_k)\right\} \vee \mathcal{M}\left\{\eta_i > \bigwedge_{k=i}^{\infty}\Psi^{-1}(1-\alpha_k)\right\} \\
&= \bigvee_{i=1}^{\infty}\bigvee_{k=i}^{\infty}\alpha_k \vee \left(1 - \bigwedge_{k=i}^{\infty}(1-\alpha_k)\right) = \bigvee_{k=1}^{\infty}\alpha_k.
\end{aligned}$$

Thus we obtain

$$\mathcal{M}\{\tau \le t\} = \bigvee_{k=1}^{\infty}\alpha_k$$

and the theorem is verified.

12.5 Age Replacement Policy

Age replacement means that an element is always replaced at failure or at an age s. Assume that the lifetimes of the elements are iid uncertain variables $\xi_1, \xi_2, \cdots$ with a common uncertainty distribution Φ. Then the actual lifetimes of the elements are iid uncertain variables

$$\xi_1 \wedge s,\ \xi_2 \wedge s,\ \cdots \tag{12.29}$$

which may generate an uncertain renewal process

$$N_t = \max_{n\ge 0}\left\{n \;\Big|\; \sum_{i=1}^{n}(\xi_i \wedge s) \le t\right\}. \tag{12.30}$$

Let a denote the "failure replacement" cost of replacing an element when it fails earlier than s, and b the "planned replacement" cost of replacing an element at the age s. Note that $a > b > 0$ is always assumed. Define

$$f(x) = \begin{cases} a, & \text{if } x < s \\ b, & \text{if } x = s. \end{cases} \tag{12.31}$$

Then $f(\xi_i \wedge s)$ is just the cost of replacing the ith element, and the average replacement cost before the time t is

$$\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s). \tag{12.32}$$

Theorem 12.11 *(Yao-Ralescu [169]) Assume $\xi_1, \xi_2, \cdots$ are iid uncertain lifetimes and s is a positive number. Then*

$$\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s) \to \frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \tag{12.33}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: At first, the average replacement cost before time t may be rewritten as

$$\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s) = \frac{\displaystyle\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\displaystyle\sum_{i=1}^{N_t}(\xi_i \wedge s)} \times \frac{\displaystyle\sum_{i=1}^{N_t}(\xi_i \wedge s)}{t}. \tag{12.34}$$

For any real number x, on the one hand, we have

$$\begin{aligned}
&\left\{\sum_{i=1}^{N_t} f(\xi_i \wedge s)\Big/\sum_{i=1}^{N_t}(\xi_i \wedge s) \le x\right\}\\
&= \bigcup_{n=1}^{\infty}\left\{(N_t = n) \cap \left(\sum_{i=1}^{n} f(\xi_i \wedge s)\Big/\sum_{i=1}^{n}(\xi_i \wedge s) \le x\right)\right\}\\
&\supset \bigcup_{n=1}^{\infty}\left\{(N_t = n) \cap \bigcap_{i=1}^{n}\left(f(\xi_i \wedge s)/(\xi_i \wedge s) \le x\right)\right\}\\
&\supset \bigcup_{n=1}^{\infty}\left\{(N_t = n) \cap \bigcap_{i=1}^{\infty}\left(f(\xi_i \wedge s)/(\xi_i \wedge s) \le x\right)\right\}\\
&\supset \bigcap_{i=1}^{\infty}\left(f(\xi_i \wedge s)/(\xi_i \wedge s) \le x\right)
\end{aligned}$$

and

$$\mathcal{M}\left\{\frac{\displaystyle\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\displaystyle\sum_{i=1}^{N_t}(\xi_i \wedge s)} \le x\right\} \ge \mathcal{M}\left\{\bigcap_{i=1}^{\infty}\left(\frac{f(\xi_i \wedge s)}{\xi_i \wedge s} \le x\right)\right\} = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\}.$$

On the other hand, we have

$$
\begin{aligned}
&\left\{\sum_{i=1}^{N_t} f(\xi_i \wedge s)/\sum_{i=1}^{N_t}(\xi_i \wedge s) \le x\right\}\\
&= \bigcup_{n=1}^{\infty}\left\{(N_t = n) \cap \left(\sum_{i=1}^{n} f(\xi_i \wedge s)/\sum_{i=1}^{n}(\xi_i \wedge s) \le x\right)\right\}\\
&\subset \bigcup_{n=1}^{\infty}\left\{(N_t = n) \cap \bigcup_{i=1}^{n} (f(\xi_i \wedge s)/(\xi_i \wedge s) \le x)\right\}\\
&\subset \bigcup_{n=1}^{\infty}\left\{(N_t = n) \cap \bigcup_{i=1}^{\infty} (f(\xi_i \wedge s)/(\xi_i \wedge s) \le x)\right\}\\
&\subset \bigcup_{i=1}^{\infty} (f(\xi_i \wedge s)/(\xi_i \wedge s) \le x)
\end{aligned}
$$

and

$$
\mathcal{M}\left\{\frac{\displaystyle\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\displaystyle\sum_{i=1}^{N_t}(\xi_i \wedge s)} \le x\right\} \le \mathcal{M}\left\{\bigcup_{i=1}^{\infty}\left(\frac{f(\xi_i \wedge s)}{\xi_i \wedge s} \le x\right)\right\} = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\}.
$$

Thus for any real number x, we have

$$
\mathcal{M}\left\{\frac{\displaystyle\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\displaystyle\sum_{i=1}^{N_t}(\xi_i \wedge s)} \le x\right\} = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\}.
$$

Hence

$$
\frac{\displaystyle\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\displaystyle\sum_{i=1}^{N_t}(\xi_i \wedge s)} \quad \text{and} \quad \frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s}
$$

are identically distributed uncertain variables. Since

$$
\frac{\displaystyle\sum_{i=1}^{N_t}(\xi_i \wedge s)}{t} \to 1
$$

as $t \to \infty$, it follows from (12.34) that (12.33) holds. The theorem is verified.

Theorem 12.12 *(Yao-Ralescu [169]) Assume* $\xi_1, \xi_2, \cdots$ *are iid uncertain lifetimes with a common continuous uncertainty distribution* Φ*, and* s *is a positive number. Then the long-run average replacement cost is*

$$\lim_{t\to\infty} E\left[\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s)\right] = \frac{b}{s} + \frac{a-b}{s}\Phi(s) + a\int_0^s \frac{\Phi(x)}{x^2}\mathrm{d}x. \tag{12.35}$$

Proof: Let $\Psi(x)$ be the uncertainty distribution of $f(\xi_1 \wedge s)/(\xi_1 \wedge s)$. It follows from (12.31) that $f(\xi_1 \wedge s) \ge b$ and $\xi_1 \wedge s \le s$. Thus we have

$$\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \ge \frac{b}{s}$$

almost surely. If $x < b/s$, then

$$\Psi(x) = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\} = 0.$$

If $b/s \le x < a/s$, then

$$\Psi(x) = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\} = \mathcal{M}\{\xi_1 \ge s\} = 1 - \Phi(s).$$

If $x \ge a/s$, then

$$\Psi(x) = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\} = \mathcal{M}\left\{\frac{a}{\xi_1} \le x\right\} = \mathcal{M}\left\{\xi_1 \ge \frac{a}{x}\right\} = 1 - \Phi\left(\frac{a}{x}\right).$$

Hence we have

$$\Psi(x) = \begin{cases} 0, & \text{if } x < b/s \\ 1 - \Phi(s), & \text{if } b/s \le x < a/s \\ 1 - \Phi(a/x), & \text{if } x \ge a/s \end{cases}$$

and

$$E\left[\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s}\right] = \int_0^{+\infty}(1 - \Psi(x))\mathrm{d}x = \frac{b}{s} + \frac{a-b}{s}\Phi(s) + a\int_0^s \frac{\Phi(x)}{x^2}\mathrm{d}x.$$

Since

$$\frac{\displaystyle\sum_{i=1}^{N_t}(\xi_i \wedge s)}{t} \le 1,$$

it follows from (12.34) that

$$\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s) \le x\right\} \ge \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi \wedge s} \le x\right\}$$

for any real number x. By using the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\lim_{t\to\infty} E\left[\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i\wedge s)\right] &= \lim_{t\to\infty}\int_0^{+\infty}\left(1-\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i\wedge s)\le x\right\}\right)\mathrm{d}x\\ &= \int_0^{+\infty}\left(1-\mathcal{M}\left\{\frac{f(\xi_1\wedge s)}{\xi_1\wedge s}\le x\right\}\right)\mathrm{d}x\\ &= E\left[\frac{f(\xi_1\wedge s)}{\xi_1\wedge s}\right].\end{aligned}$$

Hence the theorem is proved.

What is the optimal age s?

When the age replacement policy is accepted, one problem is to find the optimal age s such that the average replacement cost is minimized. That is, the optimal age s should solve

$$\min_{s\ge 0}\left(\frac{b}{s}+\frac{a-b}{s}\Phi(s)+a\int_0^s\frac{\Phi(x)}{x^2}\mathrm{d}x\right). \tag{12.36}$$

12.6 Alternating Renewal Process

Let $(\xi_1,\eta_1),(\xi_2,\eta_2),\cdots$ be a sequence of pairs of uncertain variables. We shall interpret ξ_i as the "on-times" and η_i as the "off-times" for $i=1,2,\cdots$, respectively. In this case, the i-th cycle consists of an on-time ξ_i followed by an off-time η_i.

Definition 12.5 *(Yao-Li [166]) Let $\xi_1,\xi_2,\cdots$ be iid uncertain on-times, and let $\eta_1,\eta_2,\cdots$ be iid uncertain off-times. Then*

$$A_t=\begin{cases} t-\displaystyle\sum_{i=1}^{N_t}\eta_i, & \text{if } \displaystyle\sum_{i=1}^{N_t}(\xi_i+\eta_i)\le t<\sum_{i=1}^{N_t}(\xi_i+\eta_i)+\xi_{N_t+1}\\ \displaystyle\sum_{i=1}^{N_t+1}\xi_i, & \text{if } \displaystyle\sum_{i=1}^{N_t}(\xi_i+\eta_i)+\xi_{N_t+1}\le t<\sum_{i=1}^{N_t+1}(\xi_i+\eta_i)\end{cases} \tag{12.37}$$

is called an alternating renewal process, where N_t is the renewal process with uncertain interarrival times $\xi_1+\eta_1,\xi_2+\eta_2,\cdots$

Note that the alternating renewal process A_t is just the total time at which the system is on up to time t. It is clear that

$$\sum_{i=1}^{N_t}\xi_i\le A_t\le\sum_{i=1}^{N_t+1}\xi_i \tag{12.38}$$

for each time t. We are interested in the limit property of the rate at which the system is on.

Theorem 12.13 *(Yao-Li [166], Alternating Renewal Theorem) Let A_t be an alternating renewal process with iid uncertain on-times $\xi_1, \xi_2, \cdots$ and iid uncertain off-times $\eta_1, \eta_2, \cdots$ Assume $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors. Then the availability rate*

$$\frac{A_t}{t} \to \frac{\xi_1}{\xi_1 + \eta_1} \tag{12.39}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: Write the uncertainty distributions of ξ_1 and η_1 by Φ and Ψ, respectively. Then the uncertainty distribution of $\xi_1/(\xi_1 + \eta_1)$ is

$$\Upsilon(x) = \sup_{y>0} \Phi(xy) \wedge (1 - \Psi(y - xy)). \tag{12.40}$$

On the one hand, we have

$$\begin{aligned}
&\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t} \xi_i \le x\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty} (N_t = k) \cap \left(\frac{1}{t}\sum_{i=1}^{k} \xi_i \le x\right)\right\} \\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty} \left(\sum_{i=1}^{k+1} (\xi_i + \eta_i) > t\right) \cap \left(\frac{1}{t}\sum_{i=1}^{k} \xi_i \le x\right)\right\} \\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty} \left(tx + \xi_{k+1} + \sum_{i=1}^{k+1} \eta_i > t\right) \cap \left(\frac{1}{t}\sum_{i=1}^{k} \xi_i \le x\right)\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty} \left(\frac{\xi_{k+1}}{t} + \frac{1}{t}\sum_{i=1}^{k+1} \eta_i > 1 - x\right) \cap \left(\frac{1}{t}\sum_{i=1}^{k} \xi_i \le x\right)\right\}.
\end{aligned}$$

Since

$$\frac{\xi_{k+1}}{t} \to 0, \quad \text{as } t \to \infty$$

and

$$\sum_{i=1}^{k+1} \eta_i \sim (k+1)\eta_1, \quad \sum_{i=1}^{k} \xi_i \sim k\xi_1,$$

we have

$$\begin{aligned}
&\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\xi_i\le x\right\}\\
\le\ &\lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\eta_1>\frac{t(1-x)}{k+1}\right)\cap\left(\xi_1\le\frac{tx}{k}\right)\right\}\\
=\ &\lim_{t\to\infty}\sup_{k\ge 0}\mathcal{M}\left\{\eta_1>\frac{t(1-x)}{k+1}\right\}\wedge\mathcal{M}\left\{\xi_1\le\frac{tx}{k}\right\}\\
=\ &\lim_{t\to\infty}\sup_{k\ge 0}\left(1-\Psi\left(\frac{t(1-x)}{k+1}\right)\right)\wedge\Phi\left(\frac{tx}{k}\right)\\
=\ &\sup_{y>0}\Phi(xy)\wedge(1-\Psi(y-xy))=\Upsilon(x).
\end{aligned}$$

That is,

$$\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\xi_i\le x\right\}\le\Upsilon(x). \tag{12.41}$$

On the other hand, we have

$$\begin{aligned}
&\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i>x\right\}\\
=\ &\mathcal{M}\left\{\bigcup_{k=0}^{\infty}(N_t=k)\cap\left(\frac{1}{t}\sum_{i=1}^{k+1}\xi_i>x\right)\right\}\\
\le\ &\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}(\xi_i+\eta_i)\le t\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k+1}\xi_i>x\right)\right\}\\
\le\ &\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(tx-\xi_{k+1}+\sum_{i=1}^{k}\eta_i\le t\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k+1}\xi_i>x\right)\right\}\\
=\ &\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\frac{1}{t}\sum_{i=1}^{k}\eta_i-\frac{\xi_{k+1}}{t}\le 1-x\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k+1}\xi_i>x\right)\right\}.
\end{aligned}$$

Since

$$\frac{\xi_{k+1}}{t}\to 0,\quad \text{as } t\to\infty$$

and

$$\sum_{i=1}^{k}\eta_i\sim k\eta_1,\quad \sum_{i=1}^{k+1}\xi_i\sim(k+1)\xi_1,$$

we have

$$\begin{aligned}
&\lim_{t\to\infty} \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i > x\right\} \\
&\le \lim_{t\to\infty} \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\eta_1 \le \frac{t(1-x)}{k}\right)\cap\left(\xi_1 > \frac{tx}{k+1}\right)\right\} \\
&= \lim_{t\to\infty}\sup_{k\ge 0}\mathcal{M}\left\{\eta_1 \le \frac{t(1-x)}{k}\right\}\wedge\mathcal{M}\left\{\xi_1 > \frac{tx}{k+1}\right\} \\
&= \lim_{t\to\infty}\sup_{k\ge 0}\Psi\left(\frac{t(1-x)}{k+1}\right)\wedge\left(1-\Phi\left(\frac{tx}{k+1}\right)\right) \\
&= \sup_{y>0}(1-\Phi(xy))\wedge\Psi(y-xy).
\end{aligned}$$

By using the duality of uncertain measure, we get

$$\begin{aligned}
\lim_{t\to\infty} \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i \le x\right\} &\ge 1-\sup_{y>0}(1-\Phi(xy))\wedge\Psi(y-xy) \\
&= \inf_{y>0}\Phi(xy)\vee(1-\Psi(y-xy)) = \Upsilon(x).
\end{aligned}$$

That is,

$$\lim_{t\to\infty} \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i \le x\right\} \ge \Upsilon(x). \tag{12.42}$$

Since

$$\frac{1}{t}\sum_{i=1}^{N_t}\xi_i \le \frac{A_t}{t} \le \frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i,$$

we obtain

$$\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\xi_i \le x\right\} \ge \mathcal{M}\left\{\frac{A_t}{t} \le x\right\} \ge \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i \le x\right\}.$$

It follows from (12.41) and (12.42) that for any real number x, we have

$$\lim_{t\to\infty}\left\{\frac{A_t}{t} \le x\right\} = \Upsilon(x).$$

Hence the availability rate A_t/t converges in distribution to $\xi_1/(\xi_1+\eta_1)$. The theorem is proved.

Theorem 12.14 *(Yao-Li [166], Alternating Renewal Theorem) Let A_t be an alternating renewal process with iid uncertain on-times $\xi_1, \xi_2, \cdots$ and iid* *uncertain off-times* $\eta_1, \eta_2, \cdots$ *Assume* $(\xi_1, \xi_2, \cdots)$ *and* $(\eta_1, \eta_2, \cdots)$ *are independent uncertain vectors. Then*

$$\lim_{t\to\infty} \frac{E[A_t]}{t} = E\left[\frac{\xi_1}{\xi_1 + \eta_1}\right]. \tag{12.43}$$

If those on-times and off-times have regular uncertainty distributions Φ *and* Ψ*, respectively, then*

$$\lim_{t\to\infty} \frac{E[A_t]}{t} = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi^{-1}(1-\alpha)} \mathrm{d}\alpha. \tag{12.44}$$

Proof: Write the uncertainty distributions of A_t/t and $\xi_1/(\xi_1+\eta_1)$ by $F_t(x)$ and $G(x)$, respectively. Since A_t/t converges in distribution to $\xi_1/(\xi_1+\eta_1)$, we have $F_t(x) \to G(x)$ as $t \to \infty$. It follows from the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} \frac{E[A_t]}{t} = \lim_{t\to\infty} \int_0^1 (1 - F_t(x))\mathrm{d}x = \int_0^1 (1 - G(x))\mathrm{d}x = E\left[\frac{\xi_1}{\xi_1 + \eta_1}\right].$$

Finally, since the uncertain variable $\xi_1/(\xi_1+\eta_1)$ is strictly increasing with respect to ξ_1 and strictly decreasing with respect to η_1, it has an inverse uncertainty distribution

$$G^{-1}(\alpha) = \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi(1-\alpha)}.$$

The equation (12.44) is thus obtained.

12.7 Bibliographic Notes

Uncertain renewal process was first proposed by Liu [77] in 2008. Two years later, Liu [83] proved some elementary renewal theorems for determining the average renewal number. Liu [83] also provided uncertain renewal reward process and verified some renewal reward theorems for determining the long-run reward rate. In addition, Yao-Li [166] presented uncertain alternating renewal process and proved some alternating renewal theorems for determining the availability rate.

Based on the theory of uncertain renewal process, Liu [89] presented an uncertain insurance model by assuming the claim is an uncertain renewal reward process, and proved a formula for calculating ruin index. In addition, Yao [182] derived the uncertainty distribution of ruin time. Furthermore, Ke-Yao [66] and Zhang-Guo [195] discussed the uncertain block replacement policy, and Yao-Ralescu [169] investigated the uncertain age replacement policy and obtained the long-run average replacement cost.

Chapter 13

Uncertain Calculus

Uncertain calculus is a branch of mathematics that deals with differentiation and integration of uncertain processes. This chapter will introduce Liu process, Liu integral, fundamental theorem, chain rule, change of variables, and integration by parts.

13.1 Liu Process

In 2009, Liu [79] investigated a type of stationary independent increment process whose increments are normal uncertain variables. Later, this process was named by the academic community as Liu process due to its importance and usefulness. A formal definition is given below.

Definition 13.1 *(Liu [79]) An uncertain process C_t is said to be a Liu process if*
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

It is clear that a Liu process C_t is a stationary independent increment process and has a normal uncertainty distribution with expected value 0 and variance t^2. The uncertainty distribution of C_t is

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1} \tag{13.1}$$

and inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = \frac{t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \tag{13.2}$$

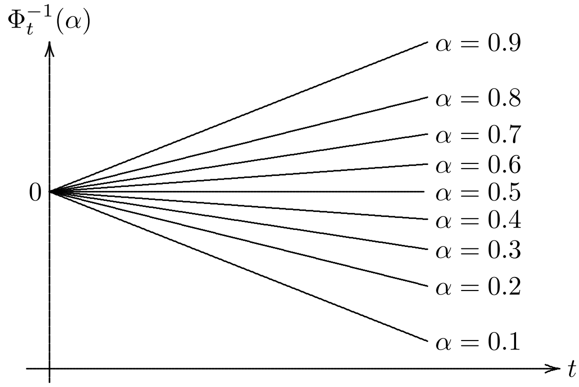


Figure 13.1: Inverse Uncertainty Distribution of Liu Process

that are homogeneous linear functions of time t for any given α. See Figure 13.1.

A Liu process is described by three properties in the above definition. Does such an uncertain process exist? The following theorem will answer this question.

Theorem 13.1 *(Liu [83], Existence Theorem) There exists a Liu process.*

Proof: It follows from Theorem 11.19 that there exists a stationary independent increment process C_t whose inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} t.$$

Furthermore, C_t has a Lipschitz continuous version. It is also easy to verify that every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2. Hence there exists a Liu process.

Theorem 13.2 *Let C_t be a Liu process. Then for each time $t > 0$, the ratio C_t/t is a normal uncertain variable with expected value 0 and variance 1. That is,*

$$\frac{C_t}{t} \sim \mathcal{N}(0, 1) \tag{13.3}$$

for any $t > 0$.

Proof: Since C_t is a normal uncertain variable $\mathcal{N}(0, t)$, the operational law tells us that C_t/t has an uncertainty distribution

$$\Psi(x) = \Phi_t(tx) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}}\right)\right)^{-1}.$$

Hence C_t/t is a normal uncertain variable with expected value 0 and variance 1. The theorem is verified.

Theorem 13.3 *(Liu [83]) Let C_t be a Liu process. Then for each time t, we have*

$$\frac{t^2}{2} \le E[C_t^2] \le t^2. \tag{13.4}$$

Proof: Note that C_t is a normal uncertain variable and has an uncertainty distribution $\Phi_t(x)$ in (13.1). It follows from the definition of expected value that

$$E[C_t^2] = \int_0^{+\infty} \mathcal{M}\{C_t^2 \ge x\}\mathrm{d}x = \int_0^{+\infty} \mathcal{M}\{(C_t \ge \sqrt{x}) \cup (C_t \le -\sqrt{x})\}\mathrm{d}x.$$

On the one hand, we have

$$\begin{aligned} E[C_t^2] &\le \int_0^{+\infty} (\mathcal{M}\{C_t \ge \sqrt{x}\} + \mathcal{M}\{C_t \le -\sqrt{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi_t(\sqrt{x}) + \Phi_t(-\sqrt{x}))\mathrm{d}x = t^2. \end{aligned}$$

On the other hand, we have

$$E[C_t^2] \ge \int_0^{+\infty} \mathcal{M}\{C_t \ge \sqrt{x}\}\mathrm{d}x = \int_0^{+\infty} (1 - \Phi_t(\sqrt{x}))\mathrm{d}x = \frac{t^2}{2}.$$

Hence (13.4) is proved.

Theorem 13.4 *(Iwamura-Xu [58]) Let C_t be a Liu process. Then for each time t, we have*

$$1.24t^4 < V[C_t^2] < 4.31t^4. \tag{13.5}$$

Proof: Let q be the expected value of C_t^2. On the one hand, it follows from the definition of variance that

$$\begin{aligned} V[C_t^2] &= \int_0^{+\infty} \mathcal{M}\{(C_t^2 - q)^2 \ge x\}\mathrm{d}x \\ &\le \int_0^{+\infty} \mathcal{M}\left\{C_t \ge \sqrt{q + \sqrt{x}}\right\}\mathrm{d}x \\ &\quad + \int_0^{+\infty} \mathcal{M}\left\{C_t \le -\sqrt{q + \sqrt{x}}\right\}\mathrm{d}x \\ &\quad\quad + \int_0^{+\infty} \mathcal{M}\left\{-\sqrt{q - \sqrt{x}} \le C_t \le \sqrt{q - \sqrt{x}}\right\}\mathrm{d}x. \end{aligned}$$

Since $t^2/2 \le q \le t^2$, we have

$$\begin{aligned}
\text{First Term} &= \int_0^{+\infty} \mathcal{M}\left\{C_t \ge \sqrt{q+\sqrt{x}}\right\} \mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{C_t \ge \sqrt{t^2/2+\sqrt{x}}\right\} \mathrm{d}x \\
&= \int_0^{+\infty} \left(1 - \left(1+\exp\left(-\frac{\pi\sqrt{t^2/2+\sqrt{x}}}{\sqrt{3}t}\right)\right)^{-1}\right) \mathrm{d}x \\
&\le 1.725t^4,
\end{aligned}$$

$$\begin{aligned}
\text{Second Term} &= \int_0^{+\infty} \mathcal{M}\left\{C_t \le -\sqrt{q+\sqrt{x}}\right\} \mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{C_t \le -\sqrt{t^2/2+\sqrt{x}}\right\} \mathrm{d}x \\
&= \int_0^{+\infty} \left(1+\exp\left(\frac{\pi\sqrt{t^2/2+\sqrt{x}}}{\sqrt{3}t}\right)\right)^{-1} \mathrm{d}x \\
&\le 1.725t^4,
\end{aligned}$$

$$\begin{aligned}
\text{Third Term} &= \int_0^{+\infty} \mathcal{M}\left\{-\sqrt{q-\sqrt{x}} \le C_t \le \sqrt{q-\sqrt{x}}\right\} \mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{C_t \le \sqrt{q-\sqrt{x}}\right\} \mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{C_t \le \sqrt{t^2-\sqrt{x}}\right\} \mathrm{d}x \\
&= \int_0^{+\infty} \left(1+\exp\left(-\frac{\pi\sqrt{t^2+\sqrt{x}}}{\sqrt{3}t}\right)\right)^{-1} \mathrm{d}x \\
&< 0.86t^4.
\end{aligned}$$

It follows from the above three upper bounds that

$$V[C_t^2] < 1.725t^4 + 1.725t^4 + 0.86t^4 = 4.31t^4.$$

On the other hand, we have

$$
\begin{aligned}
V[C_t^2] &= \int_0^{+\infty} \mathcal{M}\{(C_t^2 - q)^2 \geq x\}\mathrm{d}x \\
&\geq \int_0^{+\infty} \mathcal{M}\left\{C_t \geq \sqrt{q + \sqrt{x}}\right\} \mathrm{d}x \\
&\geq \int_0^{+\infty} \mathcal{M}\left\{C_t \geq \sqrt{t^2 + \sqrt{x}}\right\} \mathrm{d}x \\
&= \int_0^{+\infty} \left(1 - \left(1 + \exp\left(-\frac{\pi\sqrt{t^2+\sqrt{x}}}{\sqrt{3}t}\right)\right)^{-1}\right) \mathrm{d}x \\
&> 1.24t^4.
\end{aligned}
$$

The theorem is thus verified. An open problem is to improve the bounds of the variance of the square of Liu process.

Definition 13.2 *Let C_t be a Liu process. Then for any real numbers e and $\sigma > 0$, the uncertain process*

$$A_t = et + \sigma C_t \tag{13.6}$$

is called an arithmetic Liu process, where e is called the drift and σ is called the diffusion.

It is clear that the arithmetic Liu process A_t is a type of stationary independent increment process. In addition, the arithmetic Liu process A_t has a normal uncertainty distribution with expected value et and variance $\sigma^2 t^2$, i.e.,

$$A_t \sim \mathcal{N}(et, \sigma t) \tag{13.7}$$

whose uncertainty distribution is

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(et - x)}{\sqrt{3}\sigma t}\right)\right)^{-1} \tag{13.8}$$

and inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{13.9}$$

Definition 13.3 *Let C_t be a Liu process. Then for any real numbers e and $\sigma > 0$, the uncertain process*

$$G_t = \exp(et + \sigma C_t) \tag{13.10}$$

is called a geometric Liu process, where e is called the log-drift and σ is called the log-diffusion.

Note that the geometric Liu process G_t has a lognormal uncertainty distribution, i.e.,

$$G_t \sim \mathcal{LOGN}(et, \sigma t) \tag{13.11}$$

whose uncertainty distribution is

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(et - \ln x)}{\sqrt{3}\sigma t}\right)\right)^{-1} \tag{13.12}$$

and inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right). \tag{13.13}$$

Furthermore, the geometric Liu process G_t has an expected value,

$$E[G_t] = \begin{cases} \sigma t\sqrt{3}\exp(et)\csc(\sigma t\sqrt{3}), & \text{if } t < \pi/(\sigma\sqrt{3}) \\ +\infty, & \text{if } t \ge \pi/(\sigma\sqrt{3}). \end{cases} \tag{13.14}$$

13.2 Liu Integral

As the most popular topic of uncertain integral, Liu integral allows us to integrate an uncertain process (the integrand) with respect to Liu process (the integrator). The result of Liu integral is another uncertain process.

Definition 13.4 *(Liu [79]) Let X_t be an uncertain process and let C_t be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|. \tag{13.15}$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}) \tag{13.16}$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Since X_t and C_t are uncertain variables at each time t, the limit in (13.16) is also an uncertain variable provided that the limit exists almost surely and is finite. Hence an uncertain process X_t is integrable with respect to C_t if and only if the limit in (13.16) is an uncertain variable.

Example 13.1: For any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from (13.16) that

$$\int_0^s \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} (C_{t_{i+1}} - C_{t_i}) \equiv C_s - C_0 = C_s.$$

That is,

$$\int_0^s \mathrm{d}C_t = C_s. \tag{13.17}$$

Example 13.2: For any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from (13.16) that

$$\begin{aligned} C_s^2 &= \sum_{i=1}^{k} \left(C_{t_{i+1}}^2 - C_{t_i}^2\right) \\ &= \sum_{i=1}^{k} \left(C_{t_{i+1}} - C_{t_i}\right)^2 + 2\sum_{i=1}^{k} C_{t_i}\left(C_{t_{i+1}} - C_{t_i}\right) \\ &\to 0 + 2\int_0^s C_t \mathrm{d}C_t \end{aligned}$$

as $\Delta \to 0$. That is,

$$\int_0^s C_t \mathrm{d}C_t = \frac{1}{2} C_s^2. \tag{13.18}$$

Example 13.3: For any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from (13.16) that

$$\begin{aligned} sC_s &= \sum_{i=1}^{k} \left(t_{i+1} C_{t_{i+1}} - t_i C_{t_i}\right) \\ &= \sum_{i=1}^{k} C_{t_{i+1}}(t_{i+1} - t_i) + \sum_{i=1}^{k} t_i (C_{t_{i+1}} - C_{t_i}) \\ &\to \int_0^s C_t \mathrm{d}t + \int_0^s t \mathrm{d}C_t \end{aligned}$$

as $\Delta \to 0$. That is,

$$\int_0^s C_t \mathrm{d}t + \int_0^s t \mathrm{d}C_t = sC_s. \tag{13.19}$$

Theorem 13.5 *If X_t is a sample-continuous uncertain process on $[a, b]$, then it is integrable with respect to C_t on $[a, b]$.*

Proof: Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a partition of the closed interval $[a, b]$. Since the uncertain process X_t is sample-continuous, almost all sample paths are continuous functions with respect to t. Hence the limit

$$\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i})$$

exists almost surely and is finite. On the other hand, since X_t and C_t are uncertain variables at each time t, the above limit is also a measurable function. Hence the limit is an uncertain variable and then X_t is integrable with respect to C_t.

Theorem 13.6 *If X_t is an integrable uncertain process on $[a, b]$, then it is integrable on each subinterval of $[a, b]$. Moreover, if $c \in [a, b]$, then*

$$\int_a^b X_t \mathrm{d}C_t = \int_a^c X_t \mathrm{d}C_t + \int_c^b X_t \mathrm{d}C_t. \tag{13.20}$$

Proof: Let $[a', b']$ be a subinterval of $[a, b]$. Since X_t is an integrable uncertain process on $[a, b]$, for any partition

$$a = t_1 < \cdots < t_m = a' < t_{m+1} < \cdots < t_n = b' < t_{n+1} < \cdots < t_{k+1} = b,$$

the limit

$$\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i})$$

exists almost surely and is finite. Thus the limit

$$\lim_{\Delta \to 0} \sum_{i=m}^{n-1} X_{t_i}(C_{t_{i+1}} - C_{t_i})$$

exists almost surely and is finite. Hence X_t is integrable on the subinterval $[a', b']$. Next, for the partition

$$a = t_1 < \cdots < t_m = c < t_{m+1} < \cdots < t_{k+1} = b,$$

we have

$$\sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}) = \sum_{i=1}^{m-1} X_{t_i}(C_{t_{i+1}} - C_{t_i}) + \sum_{i=m}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}).$$

Note that

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}),$$

$$\int_a^c X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{m-1} X_{t_i}(C_{t_{i+1}} - C_{t_i}),$$

$$\int_c^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=m}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}).$$

Hence the equation (13.20) is proved.

Theorem 13.7 *(Linearity of Liu Integral) Let X_t and Y_t be integrable uncertain processes on $[a,b]$, and let α and β be real numbers. Then*

$$\int_a^b (\alpha X_t + \beta Y_t)\mathrm{d}C_t = \alpha \int_a^b X_t \mathrm{d}C_t + \beta \int_a^b Y_t \mathrm{d}C_t. \tag{13.21}$$

Proof: Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a partition of the closed interval $[a,b]$. It follows from the definition of Liu integral that

$$\int_a^b (\alpha X_t + \beta Y_t)\mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} (\alpha X_{t_i} + \beta Y_{t_i})(C_{t_{i+1}} - C_{t_i})$$

$$= \lim_{\Delta \to 0} \alpha \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}) + \lim_{\Delta \to 0} \beta \sum_{i=1}^{k} Y_{t_i}(C_{t_{i+1}} - C_{t_i})$$

$$= \alpha \int_a^b X_t \mathrm{d}C_t + \beta \int_a^b Y_t \mathrm{d}C_t.$$

Hence the equation (13.21) is proved.

Theorem 13.8 *Let $f(t)$ be an integrable function with respect to t. Then the Liu integral*

$$\int_0^s f(t)\mathrm{d}C_t \tag{13.22}$$

is a normal uncertain variable at each time s, and

$$\int_0^s f(t)\mathrm{d}C_t \sim \mathcal{N}\left(0, \int_0^s |f(t)|\mathrm{d}t\right). \tag{13.23}$$

Proof: Since the increments of C_t are stationary and independent normal uncertain variables, for any partition of closed interval $[0,s]$ with $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from Theorem 2.11 that

$$\sum_{i=1}^{k} f(t_i)(C_{t_{i+1}} - C_{t_i}) \sim \mathcal{N}\left(0, \sum_{i=1}^{k} |f(t_i)|(t_{i+1} - t_i)\right).$$

That is, the sum is also a normal uncertain variable. Since f is an integrable function, we have

$$\sum_{i=1}^{k} |f(t_i)|(t_{i+1} - t_i) \to \int_0^s |f(t)|\mathrm{d}t$$

as the mesh $\Delta \to 0$. Hence we obtain

$$\int_0^s f(t)\mathrm{d}C_t = \lim_{\Delta\to 0}\sum_{i=1}^{k} f(t_i)(C_{t_{i+1}} - C_{t_i}) \sim \mathcal{N}\left(0, \int_0^s |f(t)|\mathrm{d}t\right).$$

The theorem is proved.

Exercise 13.1: Let s be a given time with $s > 0$. Show that the Liu integral

$$\int_0^s t\mathrm{d}C_t \tag{13.24}$$

is a normal uncertain variable $\mathcal{N}(0, s^2/2)$ and has an uncertainty distribution

$$\Phi_s(x) = \left(1 + \exp\left(-\frac{2\pi x}{\sqrt{3}s^2}\right)\right)^{-1}. \tag{13.25}$$

Exercise 13.2: For any real number α with $0 < \alpha < 1$, the uncertain process

$$F_s = \int_0^s (s-t)^{-\alpha}\mathrm{d}C_t \tag{13.26}$$

is called a *fractional Liu process* with index α. Show that F_s is a normal uncertain variable and

$$F_s \sim \mathcal{N}\left(0, \frac{s^{1-\alpha}}{1-\alpha}\right) \tag{13.27}$$

whose uncertainty distribution is

$$\Phi_s(x) = \left(1 + \exp\left(-\frac{\pi(1-\alpha)x}{\sqrt{3}s^{1-\alpha}}\right)\right)^{-1}. \tag{13.28}$$

Definition 13.5 *(Chen-Ralescu [13]) Let C_t be a Liu process and let Z_t be an uncertain process. If there exist uncertain processes μ_t and σ_t such that*

$$Z_t = Z_0 + \int_0^t \mu_s\mathrm{d}s + \int_0^t \sigma_s\mathrm{d}C_s \tag{13.29}$$

for any $t \ge 0$, then Z_t is called a general Liu process with drift μ_t and diffusion σ_t. Furthermore, Z_t has an uncertain differential

$$\mathrm{d}Z_t = \mu_t\mathrm{d}t + \sigma_t\mathrm{d}C_t. \tag{13.30}$$

Example 13.4: It follows from the equation (13.17) that Liu process C_t can be written as

$$C_t = \int_0^t \mathrm{d}C_s.$$

Thus C_t is a general Liu process with drift 0 and diffusion 1, and has an uncertain differential $\mathrm{d}C_t$.

Example 13.5: It follows from the equation (13.18) that C_t^2 can be written as

$$C_t^2 = 2\int_0^t C_s \mathrm{d}C_s.$$

Thus C_t^2 is a general Liu process with drift 0 and diffusion $2C_t$, and has an uncertain differential

$$\mathrm{d}(C_t^2) = 2C_t \mathrm{d}C_t.$$

Example 13.6: It follows from the equation (13.19) that tC_t can be written as

$$tC_t = \int_0^t C_s \mathrm{d}s + \int_0^t s \mathrm{d}C_s.$$

Thus tC_t is a general Liu process with drift C_t and diffusion t, and has an uncertain differential

$$\mathrm{d}(tC_t) = C_t \mathrm{d}t + t\mathrm{d}C_t.$$

Theorem 13.9 *(Chen-Ralescu [13]) Any general Liu process is a sample-continuous uncertain process.*

Proof: Let Z_t be a general Liu process with drift μ_t and diffusion σ_t. Then we immediately have

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_s.$$

For each $\gamma \in \Gamma$, it is obvious that

$$|Z_t(\gamma) - Z_r(\gamma)| = \left| \int_r^t \mu_s(\gamma)\mathrm{d}s + \int_r^t \sigma_s(\gamma)\mathrm{d}C_s(\gamma) \right| \to 0$$

as $r \to t$. Hence Z_t is sample-continuous and the theorem is proved.

13.3 Fundamental Theorem

Theorem 13.10 *(Liu [79], Fundamental Theorem of Uncertain Calculus) Let $h(t, c)$ be a continuously differentiable function. Then $Z_t = h(t, C_t)$ is a general Liu process and has an uncertain differential*

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t. \tag{13.31}$$

Proof: Write $\Delta C_t = C_{t+\Delta t} - C_t = C_{\Delta t}$. It follows from Theorems 13.3 and 13.4 that Δt and ΔC_t are infinitesimals with the same order. Since the function h is continuously differentiable, by using Taylor series expansion, the infinitesimal increment of Z_t has a first-order approximation,

$$\Delta Z_t = \frac{\partial h}{\partial t}(t, C_t)\Delta t + \frac{\partial h}{\partial c}(t, C_t)\Delta C_t.$$

Hence we obtain the uncertain differential (13.31) because it makes

$$Z_s = Z_0 + \int_0^s \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \int_0^s \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t. \tag{13.32}$$

This formula is an integral form of the fundamental theorem.

Example 13.7: Let us calculate the uncertain differential of tC_t. In this case, we have $h(t, c) = tc$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t, c) = c, \quad \frac{\partial h}{\partial c}(t, c) = t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}(tC_t) = C_t\mathrm{d}t + t\mathrm{d}C_t. \tag{13.33}$$

Thus tC_t is a general Liu process with drift C_t and diffusion t.

Example 13.8: Let us calculate the uncertain differential of the arithmetic Liu process $A_t = et + \sigma C_t$. In this case, we have $h(t, c) = et + \sigma c$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t, c) = e, \quad \frac{\partial h}{\partial c}(t, c) = \sigma.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}A_t = e\mathrm{d}t + \sigma\mathrm{d}C_t. \tag{13.34}$$

Thus A_t is a general Liu process with drift e and diffusion σ.

Example 13.9: Let us calculate the uncertain differential of the geometric Liu process $G_t = \exp(et + \sigma C_t)$. In this case, we have $h(t, c) = \exp(et + \sigma c)$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t, c) = eh(t, c), \quad \frac{\partial h}{\partial c}(t, c) = \sigma h(t, c).$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}G_t = eG_t\mathrm{d}t + \sigma G_t\mathrm{d}C_t. \tag{13.35}$$

Thus G_t is a general Liu process with drift eG_t and diffusion σG_t.

13.4 Chain Rule

Chain rule is a special case of the fundamental theorem of uncertain calculus.

Theorem 13.11 *(Liu [79], Chain Rule) Let $f(c)$ be a continuously differentiable function. Then $f(C_t)$ has an uncertain differential*

$$\mathrm{d}f(C_t) = f'(C_t)\mathrm{d}C_t. \tag{13.36}$$

Proof: Since $f(c)$ is a continuously differentiable function, we immediately have

$$\frac{\partial}{\partial t}f(c) = 0, \quad \frac{\partial}{\partial c}f(c) = f'(c).$$

It follows from the fundamental theorem of uncertain calculus that the equation (13.36) holds.

Example 13.10: Let us calculate the uncertain differential of C_t^2. In this case, we have $f(c) = c^2$ and $f'(c) = 2c$. It follows from the chain rule that

$$\mathrm{d}C_t^2 = 2C_t\mathrm{d}C_t. \tag{13.37}$$

Example 13.11: Let us calculate the uncertain differential of $\sin(C_t)$. In this case, we have $f(c) = \sin(c)$ and $f'(c) = \cos(c)$. It follows from the chain rule that

$$\mathrm{d}\sin(C_t) = \cos(C_t)\mathrm{d}C_t. \tag{13.38}$$

Example 13.12: Let us calculate the uncertain differential of $\exp(C_t)$. In this case, we have $f(c) = \exp(c)$ and $f'(c) = \exp(c)$. It follows from the chain rule that

$$\mathrm{d}\exp(C_t) = \exp(C_t)\mathrm{d}C_t. \tag{13.39}$$

13.5 Change of Variables

Theorem 13.12 *(Liu [79], Change of Variables) Let f be a continuously differentiable function. Then for any $s > 0$, we have*

$$\int_0^s f'(C_t)\mathrm{d}C_t = \int_{C_0}^{C_s} f'(c)\mathrm{d}c. \tag{13.40}$$

That is,

$$\int_0^s f'(C_t)\mathrm{d}C_t = f(C_s) - f(C_0). \tag{13.41}$$

Proof: Since f is a continuously differentiable function, it follows from the chain rule that

$$\mathrm{d}f(C_t) = f'(C_t)\mathrm{d}C_t.$$

This formula implies that

$$f(C_s) = f(C_0) + \int_0^s f'(C_t)\mathrm{d}C_t.$$

Hence the theorem is verified.

Example 13.13: Since the function $f'(c) = c$ has an antiderivative $f(c) = c^2/2$, it follows from the change of variables of integral that

$$\int_0^s C_t\mathrm{d}C_t = \frac{1}{2}C_s^2 - \frac{1}{2}C_0^2 = \frac{1}{2}C_s^2.$$

Example 13.14: Since the function $f'(c) = c^2$ has an antiderivative $f(c) = c^3/3$, it follows from the change of variables of integral that

$$\int_0^s C_t^2\mathrm{d}C_t = \frac{1}{3}C_s^3 - \frac{1}{3}C_0^3 = \frac{1}{3}C_s^3.$$

Example 13.15: Since the function $f'(c) = \exp(c)$ has an antiderivative $f(c) = \exp(c)$, it follows from the change of variables of integral that

$$\int_0^s \exp(C_t)\mathrm{d}C_t = \exp(C_s) - \exp(C_0) = \exp(C_s) - 1.$$

13.6 Integration by Parts

Theorem 13.13 *(Liu [79], Integration by Parts) Suppose X_t and Y_t are general Liu processes. Then*

$$\mathrm{d}(X_tY_t) = Y_t\mathrm{d}X_t + X_t\mathrm{d}Y_t. \tag{13.42}$$

Proof: Note that ΔX_t and ΔY_t are infinitesimals with the same order. Since the function xy is a continuously differentiable function with respect to x and y, by using Taylor series expansion, the infinitesimal increment of X_tY_t has a first-order approximation,

$$\Delta(X_tY_t) = Y_t\Delta X_t + X_t\Delta Y_t.$$

Hence we obtain the uncertain differential (13.42) because it makes

$$X_sY_s = X_0Y_0 + \int_0^s Y_t\mathrm{d}X_t + \int_0^s X_t\mathrm{d}Y_t. \tag{13.43}$$

The theorem is thus proved.

Example 13.16: In order to illustrate the integration by parts, let us calculate the uncertain differential of

$$Z_t = \exp(t)C_t^2.$$

In this case, we define

$$X_t = \exp(t), \quad Y_t = C_t^2.$$

Then

$$\mathrm{d}X_t = \exp(t)\mathrm{d}t, \quad \mathrm{d}Y_t = 2C_t\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}Z_t = \exp(t)C_t^2\mathrm{d}t + 2\exp(t)C_t\mathrm{d}C_t.$$

Example 13.17: The integration by parts may also calculate the uncertain differential of

$$Z_t = \sin(t+1)\int_0^t s\mathrm{d}C_s.$$

In this case, we define

$$X_t = \sin(t+1), \quad Y_t = \int_0^t s\mathrm{d}C_s.$$

Then

$$\mathrm{d}X_t = \cos(t+1)\mathrm{d}t, \quad \mathrm{d}Y_t = t\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}Z_t = \left(\int_0^t s\mathrm{d}C_s\right)\cos(t+1)\mathrm{d}t + \sin(t+1)t\mathrm{d}C_t.$$

Example 13.18: Let f and g be continuously differentiable functions. It is clear that

$$Z_t = f(t)g(C_t)$$

is an uncertain process. In order to calculate the uncertain differential of Z_t, we define

$$X_t = f(t), \quad Y_t = g(C_t).$$

Then

$$\mathrm{d}X_t = f'(t)\mathrm{d}t, \quad \mathrm{d}Y_t = g'(C_t)\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}Z_t = f'(t)g(C_t)\mathrm{d}t + f(t)g'(C_t)\mathrm{d}C_t.$$

13.7 Bibliographic Notes

Uncertain integral was proposed by Liu [77] in 2008 in order to integrate uncertain processes with respect to Liu process. One year later, Liu [79] presented the fundamental theorem of uncertain calculus from which the techniques of chain rule, change of variables, and integration by parts were derived.

Note that uncertain integral may also be defined with respect to other integrators. For example, Yao [165] defined an uncertain integral with respect to uncertain renewal process, and Chen [16] investigated an uncertain integral with respect to finite variation processes. Since then, the theory of uncertain calculus was well developed.

Chapter 14

Uncertain Differential Equation

Uncertain differential equation is a type of differential equation involving uncertain processes. This chapter will discuss the existence, uniqueness and stability of solutions of uncertain differential equations, and introduce Yao-Chen formula that represents the solution of an uncertain differential equation by a family of solutions of ordinary differential equations. On the basis of this formula, some formulas to calculate extreme value, first hitting time, and time integral of solution are provided. Furthermore, some numerical methods for solving general uncertain differential equations are designed.

14.1 Uncertain Differential Equation

Definition 14.1 *(Liu [77]) Suppose C_t is a Liu process, and f and g are two functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{14.1}$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfies (14.1) identically in t.

Remark 14.1: The uncertain differential equation (14.1) is equivalent to the uncertain integral equation

$$X_s = X_0 + \int_0^s f(t, X_t)\mathrm{d}t + \int_0^s g(t, X_t)\mathrm{d}C_t. \tag{14.2}$$

Theorem 14.1 *Let u_t and v_t be two integrable uncertain processes. Then the uncertain differential equation*

$$\mathrm{d}X_t = u_t\mathrm{d}t + v_t\mathrm{d}C_t \tag{14.3}$$

has a solution

$$X_t = X_0 + \int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s. \tag{14.4}$$

Proof: This theorem is essentially the definition of uncertain differential or a direct deduction of the fundamental theorem of uncertain calculus.

Example 14.1: Let a and b be real numbers. Consider the uncertain differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t. \tag{14.5}$$

It follows from Theorem 14.1 that the solution is

$$X_t = X_0 + \int_0^t a\mathrm{d}s + \int_0^t b\mathrm{d}C_s.$$

That is,

$$X_t = X_0 + at + bC_t. \tag{14.6}$$

Theorem 14.2 *Let u_t and v_t be two integrable uncertain processes. Then the uncertain differential equation*

$$\mathrm{d}X_t = u_t X_t \mathrm{d}t + v_t X_t \mathrm{d}C_t \tag{14.7}$$

has a solution

$$X_t = X_0 \exp\left(\int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s\right). \tag{14.8}$$

Proof: At first, the original uncertain differential equation is equivalent to

$$\frac{\mathrm{d}X_t}{X_t} = u_t \mathrm{d}t + v_t \mathrm{d}C_t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}\ln X_t = \frac{\mathrm{d}X_t}{X_t} = u_t \mathrm{d}t + v_t \mathrm{d}C_t$$

and then

$$\ln X_t = \ln X_0 + \int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s.$$

Therefore the uncertain differential equation has a solution (14.8).

Example 14.2: Let a and b be real numbers. Consider the uncertain differential equation

$$\mathrm{d}X_t = aX_t \mathrm{d}t + bX_t \mathrm{d}C_t. \tag{14.9}$$

It follows from Theorem 14.2 that the solution is

$$X_t = X_0 \exp\left(\int_0^t a\mathrm{d}s + \int_0^t b\mathrm{d}C_s\right).$$

That is,

$$X_t = X_0 \exp\left(at + bC_t\right). \tag{14.10}$$

Linear Uncertain Differential Equation

Theorem 14.3 *(Chen-Liu [5]) Let* $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ *be integrable uncertain processes. Then the linear uncertain differential equation*

$$\mathrm{d}X_t = (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t \tag{14.11}$$

has a solution

$$X_t = U_t\left(X_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s\right) \tag{14.12}$$

where

$$U_t = \exp\left(\int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{1s}\mathrm{d}C_s\right). \tag{14.13}$$

Proof: At first, we define two uncertain processes U_t and V_t via uncertain differential equations,

$$\mathrm{d}U_t = u_{1t}U_t\mathrm{d}t + v_{1t}U_t\mathrm{d}C_t, \quad \mathrm{d}V_t = \frac{u_{2t}}{U_t}\mathrm{d}t + \frac{v_{2t}}{U_t}\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}(U_tV_t) = V_t\mathrm{d}U_t + U_t\mathrm{d}V_t = (u_{1t}U_tV_t + u_{2t})\mathrm{d}t + (v_{1t}U_tV_t + v_{2t})\mathrm{d}C_t.$$

That is, the uncertain process $X_t = U_tV_t$ is a solution of the uncertain differential equation (14.11). Note that

$$U_t = U_0\exp\left(\int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{1s}\mathrm{d}C_s\right),$$

$$V_t = V_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s.$$

Taking $U_0 = 1$ and $V_0 = X_0$, we get the solution (14.12). The theorem is proved.

Example 14.3: Let m, a, σ be real numbers. Consider a linear uncertain differential equation

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma\mathrm{d}C_t. \tag{14.14}$$

At first, we have

$$U_t = \exp\left(\int_0^t (-a)\mathrm{d}s + \int_0^t 0\mathrm{d}C_s\right) = \exp(-at).$$

It follows from Theorem 14.3 that the solution is

$$X_t = \exp(-at)\left(X_0 + \int_0^t m\exp(as)\mathrm{d}s + \int_0^t \sigma\exp(as)\mathrm{d}C_s\right).$$

That is,

$$X_t = \frac{m}{a} + \exp(-at)\left(X_0 - \frac{m}{a}\right) + \sigma\exp(-at)\int_0^t \exp(as)\mathrm{d}C_s \tag{14.15}$$

provided that $a \neq 0$. Note that X_t is a normal uncertain variable, i.e.,

$$X_t \sim \mathcal{N}\left(\frac{m}{a} + \exp(-at)\left(X_0 - \frac{m}{a}\right),\ \frac{\sigma}{a} - \exp(-at)\frac{\sigma}{a}\right). \tag{14.16}$$

Example 14.4: Let m and σ be real numbers. Consider a linear uncertain differential equation

$$\mathrm{d}X_t = m\mathrm{d}t + \sigma X_t\mathrm{d}C_t. \tag{14.17}$$

At first, we have

$$U_t = \exp\left(\int_0^t 0\mathrm{d}s + \int_0^t \sigma\mathrm{d}C_s\right) = \exp(\sigma C_t).$$

It follows from Theorem 14.3 that the solution is

$$X_t = \exp(\sigma C_t)\left(X_0 + \int_0^t m\exp(-\sigma C_s)\mathrm{d}s + \int_0^t 0\mathrm{d}C_s\right).$$

That is,

$$X_t = \exp(\sigma C_t)\left(X_0 + m\int_0^t \exp(-\sigma C_s)\mathrm{d}s\right). \tag{14.18}$$

14.2 Analytic Methods

This section will provide two analytic methods for solving some nonlinear uncertain differential equations.

First Analytic Method

This subsection will introduce an analytic method for solving nonlinear uncertain differential equations like

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t X_t\mathrm{d}C_t \tag{14.19}$$

and

$$\mathrm{d}X_t = \alpha_t X_t\mathrm{d}t + g(t, X_t)\mathrm{d}C_t. \tag{14.20}$$

Theorem 14.4 *(Liu [104]) Let f be a function of two variables and let σ_t be an integrable uncertain process. Then the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t X_t\mathrm{d}C_t \tag{14.21}$$

has a solution

$$X_t = Y_t^{-1} Z_t \tag{14.22}$$

where

$$Y_t = \exp\left(-\int_0^t \sigma_s \mathrm{d}C_s\right) \tag{14.23}$$

and Z_t is the solution of the uncertain differential equation

$$\mathrm{d}Z_t = Y_t f(t, Y_t^{-1} Z_t)\mathrm{d}t \tag{14.24}$$

with initial value $Z_0 = X_0$.

Proof: At first, by using the chain rule, the uncertain process Y_t has an uncertain differential

$$\mathrm{d}Y_t = -\exp\left(-\int_0^t \sigma_s \mathrm{d}C_s\right)\sigma_t \mathrm{d}C_t = -Y_t\sigma_t\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}(X_tY_t) = X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t = -X_tY_t\sigma_t\mathrm{d}C_t + Y_t f(t, X_t)\mathrm{d}t + Y_t\sigma_tX_t\mathrm{d}C_t.$$

That is,

$$\mathrm{d}(X_tY_t) = Y_t f(t, X_t)\mathrm{d}t.$$

Defining $Z_t = X_tY_t$, we obtain $X_t = Y_t^{-1}Z_t$ and $\mathrm{d}Z_t = Y_t f(t, Y_t^{-1}Z_t)\mathrm{d}t$. Furthermore, since $Y_0 = 1$, the initial value Z_0 is just X_0. The theorem is thus verified.

Example 14.5: Let α and σ be real numbers with $\alpha \neq 1$. Consider the uncertain differential equation

$$\mathrm{d}X_t = X_t^\alpha \mathrm{d}t + \sigma X_t \mathrm{d}C_t. \tag{14.25}$$

At first, we have $Y_t = \exp(-\sigma C_t)$ and Z_t satisfies the uncertain differential equation,

$$\mathrm{d}Z_t = \exp(-\sigma C_t)(\exp(\sigma C_t)Z_t)^\alpha \mathrm{d}t = \exp((\alpha-1)\sigma C_t)Z_t^\alpha \mathrm{d}t.$$

Since $\alpha \neq 1$, we have

$$\mathrm{d}Z_t^{1-\alpha} = (1-\alpha)\exp((\alpha-1)\sigma C_t)\mathrm{d}t.$$

It follows from the fundamental theorem of uncertain calculus that

$$Z_t^{1-\alpha} = Z_0^{1-\alpha} + (1-\alpha)\int_0^t \exp((\alpha-1)\sigma C_s)\mathrm{d}s.$$

Since the initial value Z_0 is just X_0, we have

$$Z_t = \left(X_0^{1-\alpha} + (1-\alpha)\int_0^t \exp((\alpha-1)\sigma C_s)\mathrm{d}s\right)^{1/(1-\alpha)}.$$

Theorem 14.4 says the uncertain differential equation (14.25) has a solution $X_t = Y_t^{-1} Z_t$, i.e.,

$$X_t = \exp(\sigma C_t) \left(X_0^{1-\alpha} + (1-\alpha) \int_0^t \exp((\alpha - 1)\sigma C_s) \mathrm{d}s \right)^{1/(1-\alpha)}.$$

Theorem 14.5 *(Liu [104]) Let g be a function of two variables and let α_t be an integrable uncertain process. Then the uncertain differential equation*

$$\mathrm{d}X_t = \alpha_t X_t \mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{14.26}$$

has a solution

$$X_t = Y_t^{-1} Z_t \tag{14.27}$$

where

$$Y_t = \exp\left(- \int_0^t \alpha_s \mathrm{d}s \right) \tag{14.28}$$

and Z_t is the solution of the uncertain differential equation

$$\mathrm{d}Z_t = Y_t g(t, Y_t^{-1} Z_t)\mathrm{d}C_t \tag{14.29}$$

with initial value $Z_0 = X_0$.

Proof: At first, by using the chain rule, the uncertain process Y_t has an uncertain differential

$$\mathrm{d}Y_t = -\exp\left(- \int_0^t \alpha_s \mathrm{d}s \right) \alpha_t \mathrm{d}t = -Y_t \alpha_t \mathrm{d}t.$$

It follows from the integration by parts that

$$\mathrm{d}(X_t Y_t) = X_t \mathrm{d}Y_t + Y_t \mathrm{d}X_t = -X_t Y_t \alpha_t \mathrm{d}t + Y_t \alpha_t X_t \mathrm{d}t + Y_t g(t, X_t)\mathrm{d}C_t.$$

That is,

$$\mathrm{d}(X_t Y_t) = Y_t g(t, X_t)\mathrm{d}C_t.$$

Defining $Z_t = X_t Y_t$, we obtain $X_t = Y_t^{-1} Z_t$ and $\mathrm{d}Z_t = Y_t g(t, Y_t^{-1} Z_t)\mathrm{d}C_t$. Furthermore, since $Y_0 = 1$, the initial value Z_0 is just X_0. The theorem is thus verified.

Example 14.6: Let α and β be real numbers with $\beta \neq 1$. Consider the uncertain differential equation

$$\mathrm{d}X_t = \alpha X_t \mathrm{d}t + X_t^\beta \mathrm{d}C_t. \tag{14.30}$$

At first, we have $Y_t = \exp(-\alpha t)$ and Z_t satisfies the uncertain differential equation,

$$\mathrm{d}Z_t = \exp(-\alpha t)(\exp(\alpha t) Z_t)^\beta \mathrm{d}C_t = \exp((\beta - 1)\alpha t) Z_t^\beta \mathrm{d}C_t.$$

Since $\beta \neq 1$, we have

$$\mathrm{d}Z_t^{1-\beta} = (1-\beta)\exp((\beta-1)\alpha t)\mathrm{d}C_t.$$

It follows from the fundamental theorem of uncertain calculus that

$$Z_t^{1-\beta} = Z_0^{1-\beta} + (1-\beta)\int_0^t \exp((\beta-1)\alpha s)\mathrm{d}C_s.$$

Since the initial value Z_0 is just X_0, we have

$$Z_t = \left(X_0^{1-\beta} + (1-\beta)\int_0^t \exp((\beta-1)\alpha s)\mathrm{d}C_s\right)^{1/(1-\beta)}.$$

Theorem 14.5 says the uncertain differential equation (14.30) has a solution $X_t = Y_t^{-1}Z_t$, i.e.,

$$X_t = \exp(\alpha t)\left(X_0^{1-\beta} + (1-\beta)\int_0^t \exp((\beta-1)\alpha s)\mathrm{d}C_s\right)^{1/(1-\beta)}.$$

Second Analytic Method

This subsection will introduce an analytic method for solving nonlinear uncertain differential equations like

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t\mathrm{d}C_t \tag{14.31}$$

and

$$\mathrm{d}X_t = \alpha_t\mathrm{d}t + g(t, X_t)\mathrm{d}C_t. \tag{14.32}$$

Theorem 14.6 *(Yao [171]) Let f be a function of two variables and let σ_t be an integrable uncertain process. Then the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t\mathrm{d}C_t \tag{14.33}$$

has a solution

$$X_t = Y_t + Z_t \tag{14.34}$$

where

$$Y_t = \int_0^t \sigma_s\mathrm{d}C_s \tag{14.35}$$

and Z_t is the solution of the uncertain differential equation

$$\mathrm{d}Z_t = f(t, Y_t + Z_t)\mathrm{d}t \tag{14.36}$$

with initial value $Z_0 = X_0$.

Proof: At first, Y_t has an uncertain differential $\mathrm{d}Y_t = \sigma_t \mathrm{d}C_t$. It follows that

$$\mathrm{d}(X_t - Y_t) = \mathrm{d}X_t - \mathrm{d}Y_t = f(t, X_t)\mathrm{d}t + \sigma_t \mathrm{d}C_t - \sigma_t \mathrm{d}C_t.$$

That is,

$$\mathrm{d}(X_t - Y_t) = f(t, X_t)\mathrm{d}t.$$

Defining $Z_t = X_t - Y_t$, we obtain $X_t = Y_t + Z_t$ and $\mathrm{d}Z_t = f(t, Y_t + Z_t)\mathrm{d}t$. Furthermore, since $Y_0 = 0$, the initial value Z_0 is just X_0. The theorem is proved.

Example 14.7: Let α and σ be real numbers with $\alpha \neq 0$. Consider the uncertain differential equation

$$\mathrm{d}X_t = \alpha \exp(X_t)\mathrm{d}t + \sigma \mathrm{d}C_t. \tag{14.37}$$

At first, we have $Y_t = \sigma C_t$ and Z_t satisfies the uncertain differential equation,

$$\mathrm{d}Z_t = \alpha \exp(\sigma C_t + Z_t)\mathrm{d}t.$$

Since $\alpha \neq 0$, we have

$$\mathrm{d}\exp(-Z_t) = -\alpha \exp(\sigma C_t)\mathrm{d}t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\exp(-Z_t) = \exp(-Z_0) - \alpha \int_0^t \exp(\sigma C_s)\mathrm{d}s.$$

Since the initial value Z_0 is just X_0, we have

$$Z_t = X_0 - \ln\left(1 - \alpha \int_0^t \exp(X_0 + \sigma C_s)\mathrm{d}s\right).$$

Hence

$$X_t = X_0 + \sigma C_t - \ln\left(1 - \alpha \int_0^t \exp(X_0 + \sigma C_s)\mathrm{d}s\right).$$

Theorem 14.7 *(Yao [171]) Let g be a function of two variables and let α_t be an integrable uncertain process. Then the uncertain differential equation*

$$\mathrm{d}X_t = \alpha_t \mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{14.38}$$

has a solution

$$X_t = Y_t + Z_t \tag{14.39}$$

where

$$Y_t = \int_0^t \alpha_s \mathrm{d}s \tag{14.40}$$

and Z_t is the solution of the uncertain differential equation

$$\mathrm{d}Z_t = g(t, Y_t + Z_t)\mathrm{d}C_t \tag{14.41}$$

with initial value $Z_0 = X_0$.

Proof: The uncertain process Y_t has an uncertain differential $\mathrm{d}Y_t = \alpha_t \mathrm{d}t$. It follows that

$$\mathrm{d}(X_t - Y_t) = \mathrm{d}X_t - \mathrm{d}Y_t = \alpha_t \mathrm{d}t + g(t, X_t)\mathrm{d}C_t - \alpha_t \mathrm{d}t.$$

That is,

$$\mathrm{d}(X_t - Y_t) = g(t, X_t)\mathrm{d}C_t.$$

Defining $Z_t = X_t - Y_t$, we obtain $X_t = Y_t + Z_t$ and $\mathrm{d}Z_t = g(t, Y_t + Z_t)\mathrm{d}C_t$. Furthermore, since $Y_0 = 0$, the initial value Z_0 is just X_0. The theorem is proved.

Example 14.8: Let α and σ be real numbers with $\sigma \neq 0$. Consider the uncertain differential equation

$$\mathrm{d}X_t = \alpha \mathrm{d}t + \sigma \exp(X_t)\mathrm{d}C_t. \tag{14.42}$$

At first, we have $Y_t = \alpha t$ and Z_t satisfies the uncertain differential equation,

$$\mathrm{d}Z_t = \sigma \exp(\alpha t + Z_t)\mathrm{d}C_t.$$

Since $\sigma \neq 0$, we have

$$\mathrm{d}\exp(-Z_t) = \sigma \exp(\alpha t)\mathrm{d}C_t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\exp(-Z_t) = \exp(-Z_0) + \sigma \int_0^t \exp(\alpha s)\mathrm{d}C_s.$$

Since the initial value Z_0 is just X_0, we have

$$Z_t = X_0 - \ln\left(1 - \sigma \int_0^t \exp(X_0 + \alpha s)\mathrm{d}C_s\right).$$

Hence

$$X_t = X_0 + \alpha t - \ln\left(1 - \sigma \int_0^t \exp(X_0 + \alpha s)\mathrm{d}C_s\right).$$

14.3 Existence and Uniqueness

Theorem 14.8 *(Chen-Liu [5], Existence and Uniqueness Theorem) The uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{14.43}$$

has a unique solution if the coefficients $f(t,x)$ and $g(t,x)$ satisfy the linear growth condition

$$|f(t,x)| + |g(t,x)| \leq L(1 + |x|), \quad \forall x \in \Re, t \geq 0 \tag{14.44}$$

and Lipschitz condition

$$|f(t,x)-f(t,y)|+|g(t,x)-g(t,y)|\le L|x-y|,\quad \forall x,y\in\Re, t\ge 0 \tag{14.45}$$

for some constant L. Moreover, the solution is sample-continuous.

Proof: We first prove the existence of solution by a successive approximation method. Define $X_t^{(0)}=X_0$, and

$$X_t^{(n)}=X_0+\int_0^t f\left(s,X_s^{(n-1)}\right)\mathrm{d}s+\int_0^t g\left(s,X_s^{(n-1)}\right)\mathrm{d}C_s$$

for $n=1,2,\cdots$ and write

$$D_t^{(n)}(\gamma)=\max_{0\le s\le t}\left|X_s^{(n+1)}(\gamma)-X_s^{(n)}(\gamma)\right|$$

for each $\gamma\in\Gamma$. It follows from the linear growth condition and Lipschitz condition that

$$\begin{aligned}D_t^{(0)}(\gamma)&=\max_{0\le s\le t}\left|\int_0^s f(v,X_0)\mathrm{d}v+\int_0^s g(v,X_0)\mathrm{d}C_v(\gamma)\right|\\&\le\int_0^t|f(v,X_0)|\,\mathrm{d}v+K_\gamma\int_0^t|g(v,X_0)|\,\mathrm{d}v\\&\le(1+|X_0|)L(1+K_\gamma)t\end{aligned}$$

where K_γ is the Lipschitz constant to the sample path $C_t(\gamma)$. In fact, by using the induction method, we may verify

$$D_t^{(n)}(\gamma)\le(1+|X_0|)\frac{L^{n+1}(1+K_\gamma)^{n+1}}{(n+1)!}t^{n+1}$$

for each n. This means that, for each $\gamma\in\Gamma$, the sample paths $X_t^{(k)}(\gamma)$ converges uniformly on any given time interval. Write the limit by $X_t(\gamma)$ that is just a solution of the uncertain differential equation because

$$X_t=X_0+\int_0^t f(s,X_s)\mathrm{d}s+\int_0^t g(s,X_s)\mathrm{d}C_s.$$

Next we prove that the solution is unique. Assume that both X_t and X_t^* are solutions of the uncertain differential equation. Then for each $\gamma\in\Gamma$, it follows from the linear growth condition and Lipschitz condition that

$$|X_t(\gamma)-X_t^*(\gamma)|\le L(1+K_\gamma)\int_0^t|X_v(\gamma)-X_v^*(\gamma)|\mathrm{d}v.$$

By using Gronwall inequality, we obtain

$$|X_t(\gamma)-X_t^*(\gamma)|\le 0\cdot\exp(L(1+K_\gamma)t)=0.$$

Hence $X_t = X_t^*$. The uniqueness is verified. Finally, for each $\gamma \in \Gamma$, we have

$$|X_t(\gamma) - X_r(\gamma)| = \left| \int_r^t f(s, X_s(\gamma))\mathrm{d}s + \int_r^t g(s, X_s(\gamma))\mathrm{d}C_s(\gamma) \right| \to 0$$

as $r \to t$. Thus X_t is sample-continuous and the theorem is proved.

14.4 Stability

Definition 14.2 *(Liu [79]) An uncertain differential equation is said to be stable if for any two solutions X_t and Y_t, we have*

$$\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} = 1 \tag{14.46}$$

for any given number $\varepsilon > 0$.

Example 14.9: In order to illustrate the concept of stability, let us consider the uncertain differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t. \tag{14.47}$$

It is clear that two solutions with initial values X_0 and Y_0 are

$$X_t = X_0 + at + bC_t,$$

$$Y_t = Y_0 + at + bC_t.$$

Then for any given number $\varepsilon > 0$, we have

$$\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} = \lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_0 - Y_0| < \varepsilon\} = 1.$$

Hence the uncertain differential equation (14.47) is stable.

Example 14.10: Some uncertain differential equations are not stable. For example, consider

$$\mathrm{d}X_t = X_t\mathrm{d}t + b\mathrm{d}C_t. \tag{14.48}$$

It is clear that two solutions with different initial values X_0 and Y_0 are

$$X_t = \exp(t)X_0 + b\exp(t)\int_0^t \exp(-s)\mathrm{d}C_s,$$

$$Y_t = \exp(t)Y_0 + b\exp(t)\int_0^t \exp(-s)\mathrm{d}C_s.$$

Then for any given number $\varepsilon > 0$, we have

$$\begin{aligned} &\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} \\ =\ &\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{\exp(t)|X_0 - Y_0| < \varepsilon \text{ for all } t \geq 0\} = 0. \end{aligned}$$

Hence the uncertain differential equation (14.48) is unstable.

Theorem 14.9 *(Yao-Gao-Gao [167], Stability Theorem) The uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{14.49}$$

is stable if the coefficients $f(t,x)$ and $g(t,x)$ satisfy the linear growth condition

$$|f(t,x)| + |g(t,x)| \le K(1+|x|), \quad \forall x \in \Re, t \ge 0 \tag{14.50}$$

for some constant K and strong Lipschitz condition

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le L(t)|x-y|, \quad \forall x, y \in \Re, t \ge 0 \tag{14.51}$$

for some bounded and integrable function $L(t)$ on $[0,+\infty)$.

Proof: Since $L(t)$ is bounded on $[0,+\infty)$, there is a constant R such that $L(t) \le R$ for any t. Then the strong Lipschitz condition (14.51) implies the following Lipschitz condition,

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le R|x-y|, \quad \forall x, y \in \Re, t \ge 0. \tag{14.52}$$

It follows from the linear growth condition (14.50), the Lipschitz condition (14.52) and the existence and uniqueness theorem that the uncertain differential equation (14.49) has a unique solution. Let X_t and Y_t be two solutions with initial values X_0 and Y_0, respectively. Then for each γ, we have

$$\begin{aligned}
\mathrm{d}|X_t(\gamma) - Y_t(\gamma)| &\le |f(t, X_t(\gamma)) - f(t, Y_t(\gamma))| + |g(t, X_t(\gamma)) - g(t, Y_t(\gamma))| \\
&\le L(t)|X_t(\gamma) - Y_t(\gamma)|\mathrm{d}t + L(t)K(\gamma)|X_t(\gamma) - Y_t(\gamma)|\mathrm{d}t \\
&= L(t)(1+K(\gamma))|X_t(\gamma) - Y_t(\gamma)|\mathrm{d}t
\end{aligned}$$

where $K(\gamma)$ is the Lipschitz constant of the sample path $C_t(\gamma)$. It follows that

$$|X_t(\gamma) - Y_t(\gamma)| \le |X_0 - Y_0| \exp\left((1+K(\gamma))\int_0^{+\infty} L(s)\mathrm{d}s\right).$$

Thus for any given $\varepsilon > 0$, we always have

$$\begin{aligned}
&\mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \ge 0\} \\
\ge\ &\mathcal{M}\left\{|X_0 - Y_0| \exp\left((1+K(\gamma))\int_0^{+\infty} L(s)\mathrm{d}s\right) < \varepsilon\right\}.
\end{aligned}$$

Since

$$\mathcal{M}\left\{|X_0 - Y_0| \exp\left((1+K(\gamma))\int_0^{+\infty} L(s)\mathrm{d}s\right) < \varepsilon\right\} \to 1$$

as $|X_0 - Y_0| \to 0$, we obtain

$$\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \ge 0\} = 1.$$

Hence the uncertain differential equation is stable.

Exercise 14.1: Suppose $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ are bounded functions with respect to t such that

$$\int_0^{+\infty} |u_{1t}|\mathrm{d}t < +\infty, \quad \int_0^{+\infty} |v_{1t}|\mathrm{d}t < +\infty. \tag{14.53}$$

Show that the linear uncertain differential equation

$$\mathrm{d}X_t = (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t \tag{14.54}$$

is stable.

14.5 α-Path

Definition 14.3 *(Yao-Chen [170]) Let α be a number with $0 < \alpha < 1$. An uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{14.55}$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t \tag{14.56}$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}. \tag{14.57}$$

Remark 14.2: Note that each α-path X_t^α is a real-valued function of time t, but is not necessarily one of sample paths. Furthermore, almost all α-paths are continuous functions with respect to time t.

Example 14.11: The uncertain differential equation $\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t$ with $X_0 = 0$ has an α-path

$$X_t^\alpha = at + |b|\Phi^{-1}(\alpha)t \tag{14.58}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution.

Example 14.12: The uncertain differential equation $\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t$ with $X_0 = 1$ has an α-path

$$X_t^\alpha = \exp\left(at + |b|\Phi^{-1}(\alpha)t\right) \tag{14.59}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution.

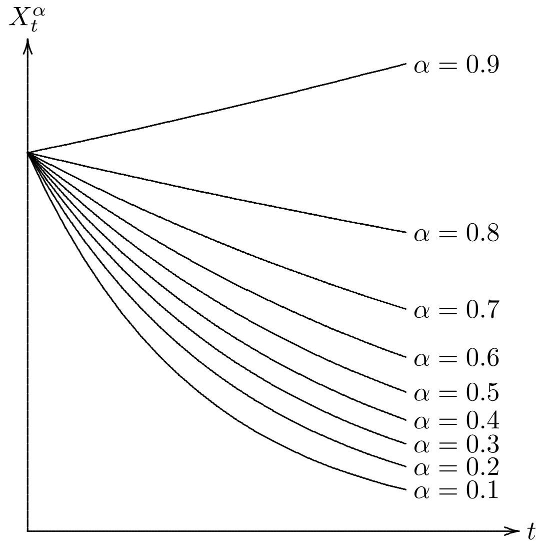


Figure 14.1: A Spectrum of α-Paths of $\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t$

14.6 Yao-Chen Formula

Yao-Chen formula relates uncertain differential equations and ordinary differential equations, just like that Feynman-Kac formula relates stochastic differential equations and partial differential equations.

Theorem 14.10 *(Yao-Chen Formula [170]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{14.60}$$

respectively. Then

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha, \tag{14.61}$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha. \tag{14.62}$$

Proof: At first, for each α-path X_t^α, we divide the time interval into two parts,

$$T^+ = \left\{t \mid g\left(t, X_t^\alpha\right) \ge 0\right\},$$

$$T^- = \left\{t \mid g\left(t, X_t^\alpha\right) < 0\right\}.$$

It is obvious that $T^+ \cap T^- = \emptyset$ and $T^+ \cup T^- = [0, +\infty)$. Write

$$\Lambda_1^+ = \left\{\gamma \mid \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \le \Phi^{-1}(\alpha) \text{ for any } t \in T^+\right\},$$

$$\Lambda_1^- = \left\{\gamma \mid \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \geq \Phi^{-1}(1-\alpha) \text{ for any } t \in T^-\right\}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution. Since T^+ and T^- are disjoint sets and C_t has independent increments, we get

$$\mathcal{M}\{\Lambda_1^+\} = \alpha, \quad \mathcal{M}\{\Lambda_1^-\} = \alpha, \quad \mathcal{M}\{\Lambda_1^+ \cap \Lambda_1^-\} = \alpha.$$

For any $\gamma \in \Lambda_1^+ \cap \Lambda_1^-$, we always have

$$g(t, X_t(\gamma))\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \leq |g(t, X_t^\alpha)|\Phi^{-1}(\alpha), \ \forall t.$$

Hence $X_t(\gamma) \leq X_t^\alpha$ for all t and

$$\mathcal{M}\{X_t \leq X_t^\alpha, \ \forall t\} \geq \mathcal{M}\{\Lambda_1^+ \cap \Lambda_1^-\} = \alpha. \tag{14.63}$$

On the other hand, let us define

$$\Lambda_2^+ = \left\{\gamma \mid \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} > \Phi^{-1}(\alpha) \text{ for any } t \in T^+\right\},$$

$$\Lambda_2^- = \left\{\gamma \mid \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} < \Phi^{-1}(1-\alpha) \text{ for any } t \in T^-\right\}.$$

Since T^+ and T^- are disjoint sets and C_t has independent increments, we obtain

$$\mathcal{M}\{\Lambda_2^+\} = 1-\alpha, \quad \mathcal{M}\{\Lambda_2^-\} = 1-\alpha, \quad \mathcal{M}\{\Lambda_2^+ \cap \Lambda_2^-\} = 1-\alpha.$$

For any $\gamma \in \Lambda_2^+ \cap \Lambda_2^-$, we always have

$$g(t, X_t(\gamma))\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} > |g(t, X_t^\alpha)|\Phi^{-1}(\alpha), \ \forall t.$$

Hence $X_t(\gamma) > X_t^\alpha$ for all t and

$$\mathcal{M}\{X_t > X_t^\alpha, \ \forall t\} \geq \mathcal{M}\{\Lambda_2^+ \cap \Lambda_2^-\} = 1-\alpha. \tag{14.64}$$

Note that $\{X_t \leq X_t^\alpha, \ \forall t\}$ and $\{X_t \not\leq X_t^\alpha, \ \forall t\}$ are opposite events with each other. By using the duality axiom, we obtain

$$\mathcal{M}\{X_t \leq X_t^\alpha, \ \forall t\} + \mathcal{M}\{X_t \not\leq X_t^\alpha, \ \forall t\} = 1.$$

It follows from $\{X_t > X_t^\alpha, \ \forall t\} \subset \{X_t \not\leq X_t^\alpha, \ \forall t\}$ and monotonicity theorem that

$$\mathcal{M}\{X_t \leq X_t^\alpha, \ \forall t\} + \mathcal{M}\{X_t > X_t^\alpha, \ \forall t\} \leq 1. \tag{14.65}$$

Thus (14.61) and (14.62) follow from (14.63), (14.64) and (14.65) immediately.

Remark 14.3: It is also showed that for any $\alpha \in (0,1)$, the following two equations are true,

$$\mathcal{M}\{X_t < X_t^\alpha, \forall t\} = \alpha, \tag{14.66}$$

$$\mathcal{M}\{X_t \geq X_t^\alpha, \forall t\} = 1 - \alpha. \tag{14.67}$$

Please mention that $\{X_t < X_t^\alpha, \forall t\}$ and $\{X_t \geq X_t^\alpha, \forall t\}$ are disjoint events but not opposite. Although it is always true that

$$\mathcal{M}\{X_t < X_t^\alpha, \forall t\} + \mathcal{M}\{X_t \geq X_t^\alpha, \forall t\} \equiv 1, \tag{14.68}$$

the union of $\{X_t < X_t^\alpha, \forall t\}$ and $\{X_t \geq X_t^\alpha, \forall t\}$ does not make the universal set, and it is possible that

$$\mathcal{M}\{(X_t < X_t^\alpha, \forall t) \cup (X_t \geq X_t^\alpha, \forall t)\} < 1. \tag{14.69}$$

Uncertainty Distribution of Solution

Theorem 14.11 *(Yao-Chen [170]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{14.70}$$

respectively. Then the solution X_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = X_t^\alpha. \tag{14.71}$$

Proof: Note that $\{X_t \leq X_t^\alpha\} \supset \{X_s \leq X_s^\alpha, \forall s\}$ holds. By using the monotonicity theorem and Yao-Chen formula, we obtain

$$\mathcal{M}\{X_t \leq X_t^\alpha\} \geq \mathcal{M}\{X_s \leq X_s^\alpha, \forall s\} = \alpha. \tag{14.72}$$

Similarly, we also have

$$\mathcal{M}\{X_t > X_t^\alpha\} \geq \mathcal{M}\{X_s > X_s^\alpha, \forall s\} = 1 - \alpha. \tag{14.73}$$

In addition, since $\{X_t \leq X_t^\alpha\}$ and $\{X_t > X_t^\alpha\}$ are opposite events, the duality axiom makes

$$\mathcal{M}\{X_t \leq X_t^\alpha\} + \mathcal{M}\{X_t > X_t^\alpha\} = 1. \tag{14.74}$$

It follows from (14.72), (14.73) and (14.74) that $\mathcal{M}\{X_t \leq X_t^\alpha\} = \alpha$. The theorem is thus verified.

Exercise 14.2: Show that the solution of the uncertain differential equation $\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t$ with $X_0 = 0$ has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = at + |b|\Phi^{-1}(\alpha)t \tag{14.75}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution.

Exercise 14.3: Show that the solution of the uncertain differential equation $\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t$ with $X_0 = 1$ has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = \exp\left(at + |b|\Phi^{-1}(\alpha)t\right) \tag{14.76}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution.

Expected Value of Solution

Theorem 14.12 *(Yao-Chen [170]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{14.77}$$

respectively. Then for any monotone (increasing or decreasing) function J, we have

$$E[J(X_t)] = \int_0^1 J(X_t^\alpha)\mathrm{d}\alpha. \tag{14.78}$$

Proof: At first, it follows from Yao-Chen formula that X_t has an uncertainty distribution $\Psi_t^{-1}(\alpha) = X_t^\alpha$. Next, we may have a monotone function become a strictly monotone function by a small perturbation. When J is a strictly increasing function, it follows from Theorem 2.8 that $J(X_t)$ has an inverse uncertainty distribution

$$\Upsilon_t^{-1}(\alpha) = J(X_t^\alpha).$$

Thus we have

$$E[J(X_t)] = \int_0^1 \Upsilon_t^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 J(X_t^\alpha)\mathrm{d}\alpha.$$

When J is a strictly decreasing function, it follows from Theorem 2.13 that $J(X_t)$ has an inverse uncertainty distribution

$$\Upsilon_t^{-1}(\alpha) = J(X_t^{1-\alpha}).$$

Thus we have

$$E[J(X_t)] = \int_0^1 \Upsilon_t^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 J(X_t^{1-\alpha})\mathrm{d}\alpha = \int_0^1 J(X_t^\alpha)\mathrm{d}\alpha.$$

The theorem is thus proved.

Exercise 14.4: Let X_t and X_t^α be the solution and α-path of some uncertain differential equation. Show that

$$E[X_t] = \int_0^1 X_t^\alpha \mathrm{d}\alpha, \tag{14.79}$$

$$E[(X_t - K)^+] = \int_0^1 (X_t^\alpha - K)^+\mathrm{d}\alpha, \tag{14.80}$$

$$E[(K - X_t)^+] = \int_0^1 (K - X_t^\alpha)^+\mathrm{d}\alpha. \tag{14.81}$$

Extreme Value of Solution

Theorem 14.13 *(Yao [168]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{14.82}$$

respectively. Then for any time $s > 0$ and strictly increasing function $J(x)$, the supremum

$$\sup_{0\le t\le s} J(X_t) \tag{14.83}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} J(X_t^\alpha); \tag{14.84}$$

and the infimum

$$\inf_{0\le t\le s} J(X_t) \tag{14.85}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \inf_{0\le t\le s} J(X_t^\alpha). \tag{14.86}$$

Proof: Since $J(x)$ is a strictly increasing function with respect to x, it is always true that

$$\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^\alpha)\right\} \supset \{X_t \le X_t^\alpha, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^\alpha)\right\} \ge \mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha. \tag{14.87}$$

Similarly, we have

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) > \sup_{0\le t\le s} J(X_t^\alpha)\right\} \ge \mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1-\alpha. \tag{14.88}$$

It follows from (14.87), (14.88) and the duality axiom that

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^\alpha)\right\} = \alpha \tag{14.89}$$

which proves (14.84). Next, it is easy to verify that

$$\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^\alpha)\right\} \supset \{X_t \le X_t^\alpha, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^\alpha)\right\} \ge \mathcal{M}\{X_t \le X_t^\alpha,\ \forall t\} = \alpha. \tag{14.90}$$

Similarly, we have

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) > \inf_{0\le t\le s} J(X_t^\alpha)\right\} \ge \mathcal{M}\{X_t > X_t^\alpha,\ \forall t\} = 1-\alpha. \tag{14.91}$$

It follows from (14.90), (14.91) and the duality axiom that

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^\alpha)\right\} = \alpha \tag{14.92}$$

which proves (14.86). The theorem is thus verified.

Exercise 14.5: Let r and K be real numbers. Show that the supremum

$$\sup_{0\le t\le s} \exp(-rt)(X_t - K)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-rt)(X_t^\alpha - K)$$

for any given time $s > 0$.

Theorem 14.14 *(Yao [168]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{14.93}$$

respectively. Then for any time $s > 0$ and strictly decreasing function $J(x)$, the supremum

$$\sup_{0\le t\le s} J(X_t) \tag{14.94}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} J(X_t^{1-\alpha}); \tag{14.95}$$

and the infimum

$$\inf_{0\le t\le s} J(X_t) \tag{14.96}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \inf_{0\le t\le s} J(X_t^{1-\alpha}). \tag{14.97}$$

Proof: Since $J(x)$ is a strictly decreasing function with respect to x, it is always true that

$$\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^{1-\alpha})\right\} \supset \{X_t \ge X_t^{1-\alpha},\ \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^{1-\alpha})\right\} \ge \mathcal{M}\{X_t \ge X_t^{1-\alpha},\ \forall t\} = \alpha. \quad (14.98)$$

Similarly, we have

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) > \sup_{0\le t\le s} J(X_t^{1-\alpha})\right\} \ge \mathcal{M}\{X_t < X_t^{1-\alpha},\ \forall t\} = 1-\alpha. \quad (14.99)$$

It follows from (14.98), (14.99) and the duality axiom that

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^{1-\alpha})\right\} = \alpha \quad (14.100)$$

which proves (14.95). Next, it is easy to verify that

$$\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^{1-\alpha})\right\} \supset \{X_t \ge X_t^{1-\alpha},\ \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^{1-\alpha})\right\} \ge \mathcal{M}\{X_t \ge X_t^{1-\alpha},\ \forall t\} = \alpha. \quad (14.101)$$

Similarly, we have

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) > \inf_{0\le t\le s} J(X_t^{1-\alpha})\right\} \ge \mathcal{M}\{X_t < X_t^{1-\alpha},\ \forall t\} = 1-\alpha. \quad (14.102)$$

It follows from (14.101), (14.102) and the duality axiom that

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^{1-\alpha})\right\} = \alpha \quad (14.103)$$

which proves (14.97). The theorem is thus verified.

Exercise 14.6: Let r and K be real numbers. Show that the supremum

$$\sup_{0\le t\le s} \exp(-rt)(K - X_t)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-rt)(K - X_t^{1-\alpha})$$

for any given time $s > 0$.

First Hitting Time of Solution

Theorem 14.15 *(Yao [168]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{14.104}$$

with an initial value X_0, respectively. Then for any given level z and strictly increasing function $J(x)$, the first hitting time τ_z that $J(X_t)$ reaches z has an uncertainty distribution

$$\Psi(s) = \begin{cases} 1 - \inf\left\{\alpha \mid \sup\limits_{0 \le t \le s} J(X_t^\alpha) \ge z\right\}, & \text{if } z > J(X_0) \\ \sup\left\{\alpha \mid \inf\limits_{0 \le t \le s} J(X_t^\alpha) \le z\right\}, & \text{if } z < J(X_0). \end{cases} \tag{14.105}$$

Proof: At first, assume $z > J(X_0)$ and write

$$\alpha_0 = \inf\left\{\alpha \mid \sup_{0 \le t \le s} J(X_t^\alpha) \ge z\right\}.$$

Then we have

$$\sup_{0 \le t \le s} J(X_t^{\alpha_0}) = z,$$

$$\{\tau_z \le s\} = \left\{\sup_{0 \le t \le s} J(X_t) \ge z\right\} \supset \{X_t \ge X_t^{\alpha_0}, \forall t\},$$

$$\{\tau_z > s\} = \left\{\sup_{0 \le t \le s} J(X_t) < z\right\} \supset \{X_t < X_t^{\alpha_0}, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\{\tau_z \le s\} \ge \mathcal{M}\{X_t \ge X_t^{\alpha_0}, \forall t\} = 1 - \alpha_0,$$

$$\mathcal{M}\{\tau_z > s\} \ge \mathcal{M}\{X_t < X_t^{\alpha_0}, \forall t\} = \alpha_0.$$

It follows from $\mathcal{M}\{\tau_z \le s\} + \mathcal{M}\{\tau_z > s\} = 1$ that $\mathcal{M}\{\tau_z \le s\} = 1 - \alpha_0$. Hence the first hitting time τ_z has an uncertainty distribution

$$\Psi(s) = \mathcal{M}\{\tau_z \le s\} = 1 - \inf\left\{\alpha \mid \sup_{0 \le t \le s} J(X_t^\alpha) \ge z\right\}.$$

Similarly, assume $z < J(X_0)$ and write

$$\alpha_0 = \sup\left\{\alpha \mid \inf_{0 \le t \le s} J(X_t^\alpha) \le z\right\}.$$

Then we have

$$\inf_{0 \le t \le s} J(X_t^{\alpha_0}) = z,$$

$$\{\tau_z \le s\} = \left\{\inf_{0\le t\le s} J(X_t) \le z\right\} \supset \{X_t \le X_t^{\alpha_0},\, \forall t\},$$

$$\{\tau_z > s\} = \left\{\inf_{0\le t\le s} J(X_t) > z\right\} \supset \{X_t > X_t^{\alpha_0},\, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\{\tau_z \le s\} \ge \mathcal{M}\{X_t \le X_t^{\alpha_0},\, \forall t\} = \alpha_0,$$

$$\mathcal{M}\{\tau_z > s\} \ge \mathcal{M}\{X_t > X_t^{\alpha_0},\, \forall t\} = 1 - \alpha_0.$$

It follows from $\mathcal{M}\{\tau_z \le s\} + \mathcal{M}\{\tau_z > s\} = 1$ that $\mathcal{M}\{\tau_z \le s\} = \alpha_0$. Hence the first hitting time τ_z has an uncertainty distribution

$$\Psi(s) = \mathcal{M}\{\tau_z \le s\} = \sup\left\{\alpha \mid \inf_{0\le t\le s} J(X_t^\alpha) \le z\right\}.$$

The theorem is verified.

Theorem 14.16 *(Yao [168]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{14.106}$$

with an initial value X_0, respectively. Then for any given level z and strictly decreasing function $J(x)$, the first hitting time τ_z that $J(X_t)$ reaches z has an uncertainty distribution

$$\Psi(s) = \begin{cases} \sup\left\{\alpha \mid \sup\limits_{0\le t\le s} J(X_t^\alpha) \ge z\right\}, & \text{if } z > J(X_0) \\ 1 - \inf\left\{\alpha \mid \inf\limits_{0\le t\le s} J(X_t^\alpha) \le z\right\}, & \text{if } z < J(X_0). \end{cases} \tag{14.107}$$

Proof: At first, assume $z > J(X_0)$ and write

$$\alpha_0 = \sup\left\{\alpha \mid \sup_{0\le t\le s} J(X_t^\alpha) \ge z\right\}.$$

Then we have

$$\sup_{0\le t\le s} J(X_t^{\alpha_0}) = z,$$

$$\{\tau_z \le s\} = \left\{\sup_{0\le t\le s} J(X_t) \ge z\right\} \supset \{X_t \le X_t^{\alpha_0},\, \forall t\},$$

$$\{\tau_z > s\} = \left\{\sup_{0\le t\le s} J(X_t) < z\right\} \supset \{X_t > X_t^{\alpha_0},\, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\{\tau_z \le s\} \ge \mathcal{M}\{X_t \le X_t^{\alpha_0}, \forall t\} = \alpha_0,$$

$$\mathcal{M}\{\tau_z > s\} \ge \mathcal{M}\{X_t > X_t^{\alpha_0}, \forall t\} = 1 - \alpha_0.$$

It follows from $\mathcal{M}\{\tau_z \le s\} + \mathcal{M}\{\tau_z > s\} = 1$ that $\mathcal{M}\{\tau_z \le s\} = \alpha_0$. Hence the first hitting time τ_z has an uncertainty distribution

$$\Psi(s) = \mathcal{M}\{\tau_z \le s\} = \sup\left\{\alpha \mid \sup_{0\le t\le s} J(X_t^\alpha) \ge z\right\}.$$

Similarly, assume $z < J(X_0)$ and write

$$\alpha_0 = \inf\left\{\alpha \mid \inf_{0\le t\le s} J(X_t^\alpha) \le z\right\}.$$

Then we have

$$\inf_{0\le t\le s} J(X_t^{\alpha_0}) = z,$$

$$\{\tau_z \le s\} = \left\{\inf_{0\le t\le s} J(X_t) \le z\right\} \supset \{X_t \ge X_t^{\alpha_0}, \forall t\},$$

$$\{\tau_z > s\} = \left\{\inf_{0\le t\le s} J(X_t) > z\right\} \supset \{X_t < X_t^{\alpha_0}, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\{\tau_z \le s\} \ge \mathcal{M}\{X_t \ge X_t^{\alpha_0}, \forall t\} = 1 - \alpha_0,$$

$$\mathcal{M}\{\tau_z > s\} \ge \mathcal{M}\{X_t < X_t^{\alpha_0}, \forall t\} = \alpha_0.$$

It follows from $\mathcal{M}\{\tau_z \le s\} + \mathcal{M}\{\tau_z > s\} = 1$ that $\mathcal{M}\{\tau_z \le s\} = 1 - \alpha_0$. Hence the first hitting time τ_z has an uncertainty distribution

$$\Psi(s) = \mathcal{M}\{\tau_z \le s\} = 1 - \inf\left\{\alpha \mid \inf_{0\le t\le s} J(X_t^\alpha) \le z\right\}.$$

The theorem is verified.

Time Integral of Solution

Theorem 14.17 *(Yao [168]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{14.108}$$

respectively. Then for any time $s > 0$ and strictly increasing function $J(x)$, the time integral

$$\int_0^s J(X_t)\mathrm{d}t \tag{14.109}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s J(X_t^\alpha)\mathrm{d}t. \tag{14.110}$$

Proof: Since $J(x)$ is a strictly increasing function with respect to x, it is always true that

$$\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^\alpha)\mathrm{d}t\right\} \supset \{J(X_t) \le J(X_t^\alpha),\ \forall t\} \supset \{X_t \le X_t^\alpha,\ \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^\alpha)\mathrm{d}t\right\} \ge \mathcal{M}\{X_t \le X_t^\alpha,\ \forall t\} = \alpha. \tag{14.111}$$

Similarly, we have

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t > \int_0^s J(X_t^\alpha)\mathrm{d}t\right\} \ge \mathcal{M}\{X_t > X_t^\alpha,\ \forall t\} = 1-\alpha. \tag{14.112}$$

It follows from (14.111), (14.112) and the duality axiom that

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^\alpha)\mathrm{d}t\right\} = \alpha. \tag{14.113}$$

The theorem is thus verified.

Exercise 14.7: Let r and K be real numbers. Show that the time integral

$$\int_0^s \exp(-rt)(X_t - K)\mathrm{d}t$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s \exp(-rt)(X_t^\alpha - K)\mathrm{d}t$$

for any given time $s > 0$.

Theorem 14.18 *(Yao [168]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{14.114}$$

respectively. Then for any time $s > 0$ and strictly decreasing function $J(x)$, the time integral

$$\int_0^s J(X_t)\mathrm{d}t \tag{14.115}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s J(X_t^{1-\alpha})\mathrm{d}t. \tag{14.116}$$

Proof: Since $J(x)$ is a strictly decreasing function with respect to x, it is always true that

$$\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^{1-\alpha})\mathrm{d}t\right\} \supset \{X_t \ge X_t^{1-\alpha},\ \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^{1-\alpha})\mathrm{d}t\right\} \ge \mathcal{M}\{X_t \ge X_t^{1-\alpha},\ \forall t\} = \alpha. \quad (14.117)$$

Similarly, we have

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t > \int_0^s J(X_t^{1-\alpha})\mathrm{d}t\right\} \ge \mathcal{M}\{X_t < X_t^{1-\alpha},\ \forall t\} = 1-\alpha. \quad (14.118)$$

It follows from (14.117), (14.118) and the duality axiom that

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^{1-\alpha})\mathrm{d}t\right\} = \alpha. \quad (14.119)$$

The theorem is thus verified.

Exercise 14.8: Let r and K be real numbers. Show that the time integral

$$\int_0^s \exp(-rt)(K - X_t)\mathrm{d}t$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s \exp(-rt)(K - X_t^{1-\alpha})\mathrm{d}t$$

for any given time $s > 0$.

14.7 Numerical Methods

It is almost impossible to find analytic solutions for general uncertain differential equations. This fact provides a motivation to design some numerical methods to solve the uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t. \quad (14.120)$$

In order to do so, a key point is to obtain a spectrum of α-paths of the uncertain differential equation. For this purpose, Yao-Chen [170] designed an Euler method:

Step 1. Fix α on $(0, 1)$.

Step 2. Solve $\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t$ by any method of ordinary differential equation and obtain the α-path X_t^α, for example, by using the recursion formula

$$X_{i+1}^\alpha = X_i^\alpha + f(t_i, X_i^\alpha)h + |g(t_i, X_i^\alpha)|\Phi^{-1}(\alpha)h \tag{14.121}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution and h is the step length.

Step 3. The α-path X_t^α is obtained.

Remark 14.4: Yang-Shen [158] designed a Runge-Kutta method that replaces the recursion formula (14.121) with

$$X_{i+1}^\alpha = X_i^\alpha + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{14.122}$$

where

$$k_1 = f(t_i, X_i^\alpha) + |g(t_i, X_i^\alpha)|\Phi^{-1}(\alpha), \tag{14.123}$$

$$k_2 = f(t_i + h/2, X_i^\alpha + hk_1/2) + |g(t_i + h/2, X_i^\alpha + hk_1/2)|\Phi^{-1}(\alpha), \tag{14.124}$$

$$k_3 = f(t_i + h/2, X_i^\alpha + hk_2/2) + |g(t_i + h/2, X_i^\alpha + hk_2/2)|\Phi^{-1}(\alpha), \tag{14.125}$$

$$k_4 = f(t_i + h, X_i^\alpha + hk_3) + |g(t_i + h, X_i^\alpha + hk_3)|\Phi^{-1}(\alpha). \tag{14.126}$$

Example 14.13: In order to illustrate the numerical method, let us consider an uncertain differential equation

$$\mathrm{d}X_t = (t - X_t)\mathrm{d}t + \sqrt{1 + X_t}\mathrm{d}C_t, \quad X_0 = 1. \tag{14.127}$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this equation successfully and obtain all α-paths of the uncertain differential equation. Furthermore, we may get

$$E[X_1] \approx 0.870. \tag{14.128}$$

Example 14.14: Now we consider a nonlinear uncertain differential equation

$$\mathrm{d}X_t = \sqrt{X_t}\mathrm{d}t + (1 - t)X_t\mathrm{d}C_t, \quad X_0 = 1. \tag{14.129}$$

Note that $(1 - t)X_t$ takes not only positive values but also negative values. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may obtain all α-paths of the uncertain differential equation. Furthermore, we may get

$$E[(X_2 - 3)^+] \approx 2.845. \tag{14.130}$$

14.8 Bibliographic Notes

The study of uncertain differential equation was pioneered by Liu [77] in 2008. This work was immediately followed upon by many researchers. Nowadays, the uncertain differential equation has achieved fruitful results in both theory and practice.

The existence and uniqueness theorem of solution of uncertain differential equation was first proved by Chen-Liu [5] under linear growth condition and Lipschitz condition. Later, the theorem was verified again by Gao [46] under local linear growth condition and local Lipschitz condition.

The first concept of stability of uncertain differential equation was presented by Liu [79], and some stability theorems were proved by Yao-Gao-Gao [167]. Following that, different types of stability of uncertain differential equations were explored, for example, stability in mean (Yao-Ke-Sheng [174]), stability in moment (Sheng-Wang [135]), stability in distribution (Yang-Ni-Zhang [160]), almost sure stability (Liu-Ke-Fei [100]), and exponential stability (Sheng-Gao [139]).

In order to solve uncertain differential equations, Chen-Liu [5] obtained an analytic solution to linear uncertain differential equations. In addition, Liu [104] and Yao [171] presented a spectrum of analytic methods to solve some special classes of nonlinear uncertain differential equations.

More importantly, Yao-Chen [170] showed that the solution of an uncertain differential equation can be represented by a family of solutions of ordinary differential equations, thus relating uncertain differential equations and ordinary differential equations. On the basis of Yao-Chen formula, Yao [168] presented some formulas to calculate extreme value, first hitting time, and time integral of solution of uncertain differential equation. Furthermore, some numerical methods for solving general uncertain differential equations were designed among others by Yao-Chen [170], Yang-Shen [158], Yang-Ralescu [157], Gao [31], and Zhang-Gao-Huang [198].

Uncertain differential equation has been successfully extended in many directions, including uncertain delay differential equation (Barbacioru [2], Ge-Zhu [49] and Liu-Fei [99]), higher-order uncertain differential equation (Yao [181]), multifactor uncertain differential equation (Li-Peng-Zhang [70]), uncertain differential equation with jumps (Yao [165]), and uncertain partial differential equation (Yang-Yao [161]).

Uncertain differential equation has been widely applied in many fields such as finance (Liu [88]), optimal control (Zhu [203]), differential game (Yang-Gao [155]), heat conduction (Yang-Yao [161]), population growth (Sheng-Gao-Zhang [141]), string vibration (Gao [36]), and spring vibration (Jia-Dai [62]).

For further explorations on the development and applications of uncertain differential equation, the interested reader may consult Yao's book [181].

Chapter 15

Uncertain Finance

This chapter will introduce uncertain stock model, uncertain interest rate model, and uncertain currency model by using the tool of uncertain differential equation.

15.1 Uncertain Stock Model

In 2009 Liu [79] first supposed that the stock price follows an uncertain differential equation and presented an *uncertain stock model* in which the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma Y_t\mathrm{d}C_t \end{cases} \tag{15.1}$$

where r is the riskless interest rate, e is the log-drift, σ is the log-diffusion, and C_t is a Liu process. Note that the bond price is $X_t = X_0\exp(rt)$ and the stock price is

$$Y_t = Y_0\exp(et + \sigma C_t) \tag{15.2}$$

whose inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = Y_0\exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right). \tag{15.3}$$

15.2 European Options

This section will price European call and put options for the financial market determined by the uncertain stock model (15.1).

European Call Option

Definition 15.1 *A European call option is a contract that gives the holder the right to buy a stock at an expiration time s for a strike price K.*

Let f_c represent the price of this contract. Then the investor pays f_c for buying the contract at time 0, and has a payoff $(Y_s - K)^+$ at time s since the option is rationally exercised if and only if $Y_s > K$. Considering the time value of money resulted from the bond, the present value of the payoff is $\exp(-rs)(Y_s - K)^+$. Thus the net return of the investor at time 0 is

$$-f_c + \exp(-rs)(Y_s - K)^+. \tag{15.4}$$

On the other hand, the bank receives f_c for selling the contract at time 0, and pays $(Y_s - K)^+$ at the expiration time s. Thus the net return of the bank at the time 0 is

$$f_c - \exp(-rs)(Y_s - K)^+. \tag{15.5}$$

The fair price of this contract should make the investor and the bank have an identical expected return (we will call it *fair price principle* hereafter), i.e.,

$$-f_c + \exp(-rs)E[(Y_s - K)^+] = f_c - \exp(-rs)E[(Y_s - K)^+]. \tag{15.6}$$

Thus $f_c = \exp(-rs)E[(Y_s - K)^+]$. That is, the European call option price is just the expected present value of the payoff.

Definition 15.2 *(Liu [79]) Assume a European call option has a strike price K and an expiration time s. Then the European call option price is*

$$f_c = \exp(-rs)E[(Y_s - K)^+]. \tag{15.7}$$

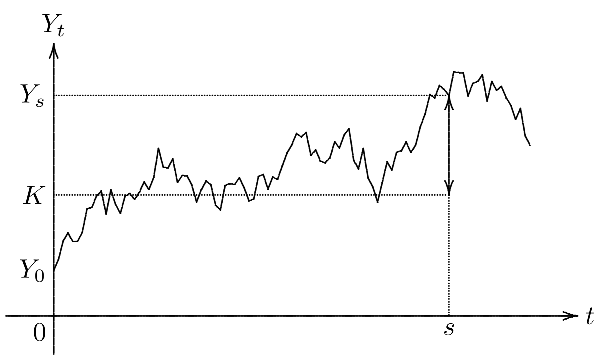


Figure 15.1: Payoff $(Y_s - K)^+$ from European Call Option

Theorem 15.1 *(Liu [79]) Assume a European call option for the uncertain stock model (15.1) has a strike price K and an expiration time s. Then the European call option price is*

$$f_c = \exp(-rs)\int_0^1 \left(Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+ \mathrm{d}\alpha. \tag{15.8}$$

Proof: Since $(Y_s - K)^+$ is an increasing function with respect to Y_s, it has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \left(Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+.$$

It follows from Definition 15.2 that the European call option price formula is just (15.8).

Remark 15.1: It is clear that the European call option price is a decreasing function of interest rate r. That is, the European call option will devaluate if the interest rate is raised; and the European call option will appreciate in value if the interest rate is reduced. In addition, the European call option price is also a decreasing function of the strike price K.

Example 15.1: Assume the interest rate $r = 0.08$, the log-drift $e = 0.06$, the log-diffusion $\sigma = 0.32$, the initial price $Y_0 = 20$, the strike price $K = 25$ and the expiration time $s = 2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the European call option price $f_c = 6.91$.

European Put Option

Definition 15.3 *A European put option is a contract that gives the holder the right to sell a stock at an expiration time s for a strike price K.*

Let f_p represent the price of this contract. Then the investor pays f_p for buying the contract at time 0, and has a payoff $(K - Y_s)^+$ at time s since the option is rationally exercised if and only if $Y_s < K$. Considering the time value of money resulted from the bond, the present value of the payoff is $\exp(-rs)(K - Y_s)^+$. Thus the net return of the investor at time 0 is

$$-f_p + \exp(-rs)(K - Y_s)^+. \tag{15.9}$$

On the other hand, the bank receives f_p for selling the contract at time 0, and pays $(K - Y_s)^+$ at the expiration time s. Thus the net return of the bank at the time 0 is

$$f_p - \exp(-rs)(K - Y_s)^+. \tag{15.10}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$-f_p + \exp(-rs)E[(K - Y_s)^+] = f_p - \exp(-rs)E[(K - Y_s)^+]. \quad (15.11)$$

Thus $f_p = \exp(-rs)E[(K - Y_s)^+]$. That is, the European put option price is just the expected present value of the payoff.

Definition 15.4 *(Liu [79]) Assume a European put option has a strike price K and an expiration time s. Then the European put option price is*

$$f_p = \exp(-rs)E[(K - Y_s)^+]. \quad (15.12)$$

Theorem 15.2 *(Liu [79]) Assume a European put option for the uncertain stock model (15.1) has a strike price K and an expiration time s. Then the European put option price is*

$$f_p = \exp(-rs)\int_0^1 \left(K - Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha. \quad (15.13)$$

Proof: Since $(K - Y_s)^+$ is a decreasing function with respect to Y_s, it has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \left(K - Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right)^+.$$

It follows from Definition 15.4 that the European put option price is

$$\begin{aligned} f_p &= \exp(-rs)\int_0^1 \left(K - Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right)^+ \mathrm{d}\alpha \\ &= \exp(-rs)\int_0^1 \left(K - Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha. \end{aligned}$$

The European put option price formula is verified.

Remark 15.2: It is easy to verify that the option price is a decreasing function of the interest rate r, and is an increasing function of the strike price K.

Example 15.2: Assume the interest rate $r = 0.08$, the log-drift $e = 0.06$, the log-diffusion $\sigma = 0.32$, the initial price $Y_0 = 20$, the strike price $K = 25$ and the expiration time $s = 2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the European put option price $f_p = 4.40$.

15.3 American Options

This section will price American call and put options for the financial market determined by the uncertain stock model (15.1).

American Call Option

Definition 15.5 *An American call option is a contract that gives the holder the right to buy a stock at any time prior to an expiration time s for a strike price K.*

Let f_c represent the price of this contract. Then the investor pays f_c for buying the contract at time 0, and has a present value of the payoff,

$$\sup_{0\le t\le s} \exp(-rt)(Y_t - K)^+. \tag{15.14}$$

Thus the net return of the investor at time 0 is

$$-f_c + \sup_{0\le t\le s} \exp(-rt)(Y_t - K)^+. \tag{15.15}$$

On the other hand, the bank receives f_c for selling the contract at time 0, and pays

$$\sup_{0\le t\le s} \exp(-rt)(Y_t - K)^+. \tag{15.16}$$

Thus the net return of the bank at the time 0 is

$$f_c - \sup_{0\le t\le s} \exp(-rt)(Y_t - K)^+. \tag{15.17}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$-f_c + E\left[\sup_{0\le t\le s} \exp(-rt)(Y_t - K)^+\right] = f_c - E\left[\sup_{0\le t\le s} \exp(-rt)(Y_t - K)^+\right].$$

Thus the American call option price is just the expected present value of the payoff.

Definition 15.6 *(Chen [6]) Assume an American call option has a strike price K and an expiration time s. Then the American call option price is*

$$f_c = E\left[\sup_{0\le t\le s} \exp(-rt)(Y_t - K)^+\right]. \tag{15.18}$$

Theorem 15.3 *(Chen [6]) Assume an American call option for the uncertain stock model (15.1) has a strike price K and an expiration time s. Then the American call option price is*

$$f_c = \int_0^1 \sup_{0\le t\le s} \exp(-rt)\left(Y_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+ \mathrm{d}\alpha.$$

Proof: It follows from Theorem 14.13 that $\sup_{0\le t\le s}\exp(-rt)(Y_t-K)^+$ has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha)=\sup_{0\le t\le s}\exp(-rt)\left(Y_0\exp\left(et+\frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K\right)^+.$$

Hence the American call option price formula follows from Definition 15.6 immediately.

Remark 15.3: It is easy to verify that the option price is a decreasing function with respect to either the interest rate r or the strike price K.

Example 15.3: Assume the interest rate $r=0.08$, the log-drift $e=0.06$, the log-diffusion $\sigma=0.32$, the initial price $Y_0=40$, the strike price $K=38$ and the expiration time $s=2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the American call option price $f_c=19.8$.

American Put Option

Definition 15.7 *An American put option is a contract that gives the holder the right to sell a stock at any time prior to an expiration time s for a strike price K.*

Let f_p represent the price of this contract. Then the investor pays f_p for buying the contract at time 0, and has a present value of the payoff,

$$\sup_{0\le t\le s}\exp(-rt)(K-Y_t)^+. \tag{15.19}$$

Thus the net return of the investor at time 0 is

$$-f_p+\sup_{0\le t\le s}\exp(-rt)(K-Y_t)^+. \tag{15.20}$$

On the other hand, the bank receives f_p for selling the contract at time 0, and pays

$$\sup_{0\le t\le s}\exp(-rt)(K-Y_t)^+. \tag{15.21}$$

Thus the net return of the bank at the time 0 is

$$f_p-\sup_{0\le t\le s}\exp(-rt)(K-Y_t)^+. \tag{15.22}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$-f_p+E\left[\sup_{0\le t\le s}\exp(-rt)(K-Y_t)^+\right]=f_p-E\left[\sup_{0\le t\le s}\exp(-rt)(K-Y_t)^+\right].$$

Thus the American put option price is just the expected present value of the payoff.

Definition 15.8 *(Chen [6]) Assume an American put option has a strike price K and an expiration time s. Then the American put option price is*

$$f_p = E\left[\sup_{0\le t\le s} \exp(-rt)(K - Y_t)^+\right]. \tag{15.23}$$

Theorem 15.4 *(Chen [6]) Assume an American put option for the uncertain stock model (15.1) has a strike price K and an expiration time s. Then the American put option price is*

$$f_p = \int_0^1 \sup_{0\le t\le s} \exp(-rt)\left(K - Y_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha.$$

Proof: It follows from Theorem 14.14 that $\sup_{0\le t\le s}\exp(-rt)(K - Y_t)^+$ has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-rt)\left(K - Y_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right)^+.$$

Hence the American put option price formula follows from Definition 15.8 immediately.

Remark 15.4: It is easy to verify that the option price is a decreasing function of the interest rate r, and is an increasing function of the strike price K.

Example 15.4: Assume the interest rate $r = 0.08$, the log-drift $e = 0.06$, the log-diffusion $\sigma = 0.32$, the initial price $Y_0 = 40$, the strike price $K = 38$ and the expiration time $s = 2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the American put option price $f_p = 3.90$.

15.4 Asian Options

This section will price Asian call and put options for the financial market determined by the uncertain stock model (15.1).

Asian Call Option

Definition 15.9 *An Asian call option is a contract whose payoff at the expiration time s is*

$$\left(\frac{1}{s}\int_0^s Y_t \mathrm{d}t - K\right)^+ \tag{15.24}$$

where K is a strike price.

Let f_c represent the price of this contract. Then the investor pays f_c for buying the contract at time 0, and has a payoff

$$\left(\frac{1}{s}\int_0^s Y_t\mathrm{d}t - K\right)^+ \tag{15.25}$$

at time s. Considering the time value of money resulted from the bond, the present value of the payoff is

$$\exp(-rs)\left(\frac{1}{s}\int_0^s Y_t\mathrm{d}t - K\right)^+. \tag{15.26}$$

Thus the net return of the investor at time 0 is

$$-f_c + \exp(-rs)\left(\frac{1}{s}\int_0^s Y_t\mathrm{d}t - K\right)^+. \tag{15.27}$$

On the other hand, the bank receives f_c for selling the contract at time 0, and pays

$$\left(\frac{1}{s}\int_0^s Y_t\mathrm{d}t - K\right)^+ \tag{15.28}$$

at the expiration time s. Thus the net return of the bank at the time 0 is

$$f_c - \exp(-rs)\left(\frac{1}{s}\int_0^s Y_t\mathrm{d}t - K\right)^+. \tag{15.29}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$\begin{aligned}-f_c + \exp(-rs)E&\left[\left(\frac{1}{s}\int_0^s Y_t\mathrm{d}t - K\right)^+\right]\\ &= f_c - \exp(-rs)E\left[\left(\frac{1}{s}\int_0^s Y_t\mathrm{d}t - K\right)^+\right].\end{aligned} \tag{15.30}$$

Thus the Asian call option price is just the expected present value of the payoff.

Definition 15.10 *(Sun-Chen [142]) Assume an Asian call option has a strike price K and an expiration time s. Then the Asian call option price is*

$$f_c = \exp(-rs)E\left[\left(\frac{1}{s}\int_0^s Y_t\mathrm{d}t - K\right)^+\right]. \tag{15.31}$$

Theorem 15.5 *(Sun-Chen [142]) Assume an Asian call option for the uncertain stock model (15.1) has a strike price K and an expiration time s. Then the Asian call option price is*

$$f_c = \exp(-rs)\int_0^1 \left(\frac{Y_0}{s}\int_0^s \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}t - K\right)^+ \mathrm{d}\alpha.$$

Proof: It follows from Theorem 14.17 that the inverse uncertainty distribution of the time integral

$$\int_0^s Y_t \mathrm{d}t$$

is

$$\Psi_s^{-1}(\alpha) = Y_0\int_0^s \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}t.$$

Hence the Asian call option price formula follows from Definition 15.10 immediately.

Asian Put Option

Definition 15.11 *An Asian put option is a contract whose payoff at the expiration time s is*

$$\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+ \tag{15.32}$$

where K is a strike price.

Let f_p represent the price of this contract. Then the investor pays f_p for buying the contract at time 0, and has a payoff

$$\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+ \tag{15.33}$$

at time s. Considering the time value of money resulted from the bond, the present value of the payoff is

$$\exp(-rs)\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+. \tag{15.34}$$

Thus the net return of the investor at time 0 is

$$-f_p + \exp(-rs)\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+. \tag{15.35}$$

On the other hand, the bank receives f_p for selling the contract at time 0, and pays

$$\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+ \tag{15.36}$$

at the expiration time s. Thus the net return of the bank at the time 0 is

$$f_p - \exp(-rs)\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+. \tag{15.37}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$\begin{aligned} &-f_p + \exp(-rs)E\left[\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+\right] \\ &= f_p - \exp(-rs)E\left[\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+\right]. \end{aligned} \tag{15.38}$$

Thus the Asian put option price should be the expected present value of the payoff.

Definition 15.12 *(Sun-Chen [142]) Assume an Asian put option has a strike price K and an expiration time s. Then the Asian put option price is*

$$f_p = \exp(-rs)E\left[\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+\right]. \tag{15.39}$$

Theorem 15.6 *(Sun-Chen [142]) Assume an Asian put option for the uncertain stock model (15.1) has a strike price K and an expiration time s. Then the Asian put option price is*

$$f_c = \exp(-rs)\int_0^1 \left(K - \frac{Y_0}{s}\int_0^s \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}t\right)^+ \mathrm{d}\alpha.$$

Proof: It follows from Theorem 14.17 that the inverse uncertainty distribution of the time integral

$$\int_0^s Y_t \mathrm{d}t$$

is

$$\Psi_s^{-1}(\alpha) = Y_0\int_0^s \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}t.$$

Hence the Asian put option price formula follows from Definition 15.12 immediately.

15.5 General Stock Model

Generally, we may assume the stock price follows a general uncertain differential equation and obtain a *general stock model* in which the bond price X_t

and the stock price Y_t are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = F(t, Y_t)\mathrm{d}t + G(t, Y_t)\mathrm{d}C_t \end{cases} \tag{15.40}$$

where r is the riskless interest rate, F and G are two functions, and C_t is a Liu process.

Theorem 15.7 *(Liu [94]) Assume a European option for the uncertain stock model (15.40) has a strike price K and an expiration time s. Then the European call option price is*

$$f_c = \exp(-rs)\int_0^1 (Y_s^\alpha - K)^+\mathrm{d}\alpha \tag{15.41}$$

and the European put option price is

$$f_p = \exp(-rs)\int_0^1 (K - Y_s^\alpha)^+\mathrm{d}\alpha \tag{15.42}$$

where Y_s^α is the α-path of the corresponding uncertain differential equation.

Proof: It follows from the fair price principle that the European call option price is

$$f_c = \exp(-rs)E[(Y_s - K)^+]. \tag{15.43}$$

By using Theorem 14.12, we get the formula (15.41). Similarly, it follows from the fair price principle that the European put option price is

$$f_p = \exp(-rs)E[(K - Y_s)^+]. \tag{15.44}$$

By using Theorem 14.12, we get the formula (15.42).

Theorem 15.8 *(Liu [94]) Assume an American option for the uncertain stock model (15.40) has a strike price K and an expiration time s. Then the American call option price is*

$$f_c = \int_0^1 \sup_{0\le t\le s} \exp(-rt)(Y_t^\alpha - K)^+\mathrm{d}\alpha \tag{15.45}$$

and the American put option price is

$$f_p = \int_0^1 \sup_{0\le t\le s} \exp(-rt)(K - Y_t^\alpha)^+\mathrm{d}\alpha \tag{15.46}$$

where Y_t^α is the α-path of the corresponding uncertain differential equation.

Proof: It follows from the fair price principle that the American call option price is

$$f_c = E\left[\sup_{0\le t\le s} \exp(-rt)(Y_t - K)^+\right]. \tag{15.47}$$

By using Theorem 14.13, we get the formula (15.45). Similarly, it follows from the fair price principle that the American put option price is

$$f_p = E\left[\sup_{0\le t\le s} \exp(-rt)(K - Y_t)^+\right]. \tag{15.48}$$

By using Theorem 14.14, we get the formula (15.46).

Theorem 15.9 *(Liu [94]) Assume an Asian option for the uncertain stock model (15.40) has a strike price K and an expiration time s. Then the Asian call option price is*

$$f_c = \exp(-rs)\int_0^1 \left(\frac{1}{s}\int_0^s Y_t^\alpha \mathrm{d}t - K\right)^+ \mathrm{d}\alpha \tag{15.49}$$

and the Asian put option price is

$$f_p = \exp(-rs)\int_0^1 \left(K - \frac{1}{s}\int_0^s Y_t^\alpha \mathrm{d}t\right)^+ \mathrm{d}\alpha \tag{15.50}$$

where Y_t^α is the α-path of the corresponding uncertain differential equation.

Proof: It follows from the fair price principle that the Asian call option price is

$$f_c = \exp(-rs)E\left[\left(\frac{1}{s}\int_0^s Y_t \mathrm{d}t - K\right)^+\right]. \tag{15.51}$$

By using Theorem 14.17, we get the formula (15.49). Similarly, it follows from the fair price principle that the Asian put option price is

$$f_p = \exp(-rs)E\left[\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+\right]. \tag{15.52}$$

By using Theorem 14.18, we get the formula (15.50).

15.6 Multifactor Stock Model

Now we assume that there are multiple stocks whose prices are determined by multiple Liu processes. In this case, we have a *multifactor stock model* in which the bond price X_t and the stock prices Y_{it} are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_{it} = e_iY_{it}\mathrm{d}t + \displaystyle\sum_{j=1}^n \sigma_{ij}Y_{it}\mathrm{d}C_{jt}, \ i = 1, 2, \cdots, m \end{cases} \tag{15.53}$$

where r is the riskless interest rate, e_i are the log-drifts, σ_{ij} are the log-diffusions, and C_{jt} are independent Liu processes, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

Portfolio Selection

For the multifactor stock model (15.53), we have the choice of $m+1$ different investments. At each time t we may choose a portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ (i.e., the investment fractions meeting $\beta_t + \beta_{1t} + \cdots + \beta_{mt} = 1$). Then the wealth Z_t at time t should follow the uncertain differential equation

$$\mathrm{d}Z_t = r\beta_t Z_t \mathrm{d}t + \sum_{i=1}^{m} e_i \beta_{it} Z_t \mathrm{d}t + \sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}\beta_{it} Z_t \mathrm{d}C_{jt}. \tag{15.54}$$

That is,

$$Z_t = Z_0 \exp(rt) \exp\left(\int_0^t \sum_{i=1}^{m}(e_i - r)\beta_{is}\mathrm{d}s + \sum_{j=1}^{n}\int_0^t \sum_{i=1}^{m} \sigma_{ij}\beta_{is}\mathrm{d}C_{js}\right).$$

Portfolio selection problem is to find an optimal portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that the wealth Z_s is maximized in the sense of expected value.

No-Arbitrage

The stock model (15.53) is said to be *no-arbitrage* if there is no portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that for some time $s > 0$, we have

$$\mathcal{M}\{\exp(-rs)Z_s \ge Z_0\} = 1 \tag{15.55}$$

and

$$\mathcal{M}\{\exp(-rs)Z_s > Z_0\} > 0 \tag{15.56}$$

where Z_t is determined by (15.54) and represents the wealth at time t.

Theorem 15.10 *(Yao's No-Arbitrage Theorem [173]) The multifactor stock model (15.53) is no-arbitrage if and only if the system of linear equations*

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} e_1 - r \\ e_2 - r \\ \vdots \\ e_m - r \end{pmatrix} \tag{15.57}$$

has a solution, i.e., $(e_1 - r, e_2 - r, \cdots, e_m - r)$ is a linear combination of column vectors $(\sigma_{11}, \sigma_{21}, \cdots, \sigma_{m1})$, $(\sigma_{12}, \sigma_{22}, \cdots, \sigma_{m2})$, $\cdots$, $(\sigma_{1n}, \sigma_{2n}, \cdots, \sigma_{mn})$.

Proof: When the portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ is accepted, the wealth at each time t is

$$Z_t = Z_0 \exp(rt) \exp\left(\int_0^t \sum_{i=1}^m (e_i - r)\beta_{is} \mathrm{d}s + \sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij}\beta_{is} \mathrm{d}C_{js}\right).$$

Thus

$$\ln(\exp(-rt)Z_t) - \ln Z_0 = \int_0^t \sum_{i=1}^m (e_i - r)\beta_{is} \mathrm{d}s + \sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij}\beta_{is} \mathrm{d}C_{js}$$

is a normal uncertain variable with expected value

$$\int_0^t \sum_{i=1}^m (e_i - r)\beta_{is} \mathrm{d}s$$

and variance

$$\left(\sum_{j=1}^n \int_0^t \left|\sum_{i=1}^m \sigma_{ij}\beta_{is}\right| \mathrm{d}s\right)^2.$$

Assume the system (15.57) has a solution. The argument breaks down into two cases. Case I: for any given time t and portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$, suppose

$$\sum_{j=1}^n \int_0^t \left|\sum_{i=1}^m \sigma_{ij}\beta_{is}\right| \mathrm{d}s = 0.$$

Then

$$\sum_{i=1}^m \sigma_{ij}\beta_{is} = 0, \quad j = 1, 2, \cdots, n,\ s \in (0, t].$$

Since the system (15.57) has a solution, we have

$$\sum_{i=1}^m (e_i - r)\beta_{is} = 0, \quad s \in (0, t]$$

and

$$\int_0^t \sum_{i=1}^m (e_i - r)\beta_{is} \mathrm{d}s = 0.$$

This fact implies that

$$\ln(\exp(-rt)Z_t) - \ln Z_0 = 0$$

and

$$\mathcal{M}\{\exp(-rt)Z_t > Z_0\} = 0.$$

That is, the stock model (15.53) is no-arbitrage. Case II: for any given time t and portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$, suppose

$$\sum_{j=1}^{n} \int_0^t \left| \sum_{i=1}^{m} \sigma_{ij}\beta_{is} \right| \mathrm{d}s \neq 0.$$

Then $\ln(\exp(-rt)Z_t) - \ln Z_0$ is a normal uncertain variable with nonzero variance and

$$\mathcal{M}\{\ln(\exp(-rt)Z_t) - \ln Z_0 \geq 0\} < 1.$$

That is,

$$\mathcal{M}\{\exp(-rt)Z_t \geq Z_0\} < 1$$

and the multifactor stock model (15.53) is no-arbitrage.

Conversely, assume the system (15.57) has no solution. Then there exist real numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that

$$\sum_{i=1}^{m} \sigma_{ij}\alpha_i = 0, \quad j = 1, 2, \cdots, n$$

and

$$\sum_{i=1}^{m} (e_i - r)\alpha_i > 0.$$

Now we take a portfolio

$$(\beta_t, \beta_{1t}, \cdots, \beta_{mt}) \equiv (1 - (\alpha_1 + \alpha_2 + \cdots + \alpha_m), \alpha_1, \alpha_2, \cdots, \alpha_m).$$

Then

$$\ln(\exp(-rt)Z_t) - \ln Z_0 = \int_0^t \sum_{i=1}^{m} (e_i - r)\alpha_i \mathrm{d}s > 0.$$

Thus we have

$$\mathcal{M}\{\exp(-rt)Z_t > Z_0\} = 1.$$

Hence the multifactor stock model (15.53) is arbitrage. The theorem is thus proved.

Theorem 15.11 *The multifactor stock model (15.53) is no-arbitrage if its log-diffusion matrix*

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mn} \end{pmatrix} \tag{15.58}$$

has rank m, i.e., the row vectors are linearly independent.

Proof: If the log-diffusion matrix (15.58) has rank m, then the system of equations (15.57) has a solution. It follows from Theorem 15.10 that the multifactor stock model (15.53) is no-arbitrage.

Theorem 15.12 *The multifactor stock model (15.53) is no-arbitrage if its log-drifts are all equal to the interest rate r, i.e.,*

$$e_i = r, \quad i = 1, 2, \cdots, m. \tag{15.59}$$

Proof: Since the log-drifts $e_i = r$ for any $i = 1, 2, \cdots, m$, we immediately have

$$(e_1 - r, e_2 - r, \cdots, e_m - r) \equiv (0, 0, \cdots, 0)$$

that is a linear combination of $(\sigma_{11}, \sigma_{21}, \cdots, \sigma_{m1})$, $(\sigma_{12}, \sigma_{22}, \cdots, \sigma_{m2})$, $\cdots$, $(\sigma_{1n}, \sigma_{2n}, \cdots, \sigma_{mn})$. It follows from Theorem 15.10 that the multifactor stock model (15.53) is no-arbitrage.

15.7 Uncertain Interest Rate Model

Real interest rates do not remain unchanged. Chen-Gao [14] assumed that the interest rate follows an uncertain differential equation and presented an uncertain interest rate model,

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma \mathrm{d}C_t \tag{15.60}$$

where m, a, σ are positive numbers. Besides, Jiao-Yao [63] investigated the uncertain interest rate model,

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}C_t. \tag{15.61}$$

More generally, we may assume the interest rate X_t follows a general uncertain differential equation and obtain a general interest rate model,

$$\mathrm{d}X_t = F(t, X_t)\mathrm{d}t + G(t, X_t)\mathrm{d}C_t \tag{15.62}$$

where F and G are two functions, and C_t is a Liu process.

Zero-Coupon Bond

A *zero-coupon bond* is a bond bought at a price lower than its *face value* that is the amount it promises to pay at the maturity date. For simplicity, we assume the face value is always 1 dollar.

Let f represent the price of this zero-coupon bond. Then the investor pays f for buying it at time 0, and receives 1 dollar at the maturity date s. Since the interest rate is X_t, the present value of 1 dollar is

$$\exp\left(-\int_0^s X_t \mathrm{d}t\right). \tag{15.63}$$

Thus the net return of the investor at time 0 is

$$-f+\exp\left(-\int_0^s X_t \mathrm{d}t\right). \tag{15.64}$$

On the other hand, the bank receives f for selling the zero-coupon bond at time 0, and pays 1 dollar at the maturity date s. Thus the net return of the bank at the time 0 is

$$f-\exp\left(-\int_0^s X_t \mathrm{d}t\right). \tag{15.65}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$-f+E\left[\exp\left(-\int_0^s X_t \mathrm{d}t\right)\right]=f-E\left[\exp\left(-\int_0^s X_t \mathrm{d}t\right)\right] \tag{15.66}$$

Thus the price of the zero-coupon bond is just the expected present value of its face value.

Definition 15.13 *(Chen-Gao [14]) Let X_t be the uncertain interest rate. Then the price of a zero-coupon bond with a maturity date s is*

$$f=E\left[\exp\left(-\int_0^s X_t \mathrm{d}t\right)\right]. \tag{15.67}$$

Theorem 15.13 *(Jiao-Yao [63]) Assume the uncertain interest rate X_t follows the uncertain differential equation (15.62). Then the price of a zero-coupon bond with maturity date s is*

$$f=\int_0^1 \exp\left(-\int_0^s X_t^\alpha \mathrm{d}t\right)\mathrm{d}\alpha \tag{15.68}$$

where X_t^α is the α-path of the corresponding uncertain differential equation.

Proof: It follows from Theorem 14.17 that the inverse uncertainty distribution of the time integral

$$\int_0^s X_t \mathrm{d}t$$

is

$$\Psi_s^{-1}(\alpha)=\int_0^s X_t^\alpha \mathrm{d}t.$$

Hence the price formula of zero-coupon bond follows from Theorem 2.26 immediately.

Interest Rate Ceiling

An interest rate ceiling is a derivative contract in which the borrower will not pay any more than a predetermined level of interest on his loan. Assume K is the maximum interest rate and s is the maturity date. For simplicity, we also assume the amount of loan is always 1 dollar.

Let f represent the price of this contract. Then the borrower pays f for buying the contract at time 0, and has a payoff

$$\exp\left(\int_0^s X_t\mathrm{d}t\right) - \exp\left(\int_0^s X_t \wedge K\mathrm{d}t\right) \tag{15.69}$$

at the maturity date s. Considering the time value of money, the present value of the payoff is

$$\begin{aligned}
&\exp\left(-\int_0^s X_t\mathrm{d}t\right)\left(\exp\left(\int_0^s X_t\mathrm{d}t\right) - \exp\left(\int_0^s X_t \wedge K\mathrm{d}t\right)\right)\\
&= 1 - \exp\left(-\int_0^s X_t\mathrm{d}t + \int_0^s X_t \wedge K\mathrm{d}t\right)\\
&= 1 - \exp\left(-\int_0^s (X_t - K)^+\mathrm{d}t\right).
\end{aligned}$$

Thus the net return of the borrower at time 0 is

$$-f + 1 - \exp\left(-\int_0^s (X_t - K)^+\mathrm{d}t\right). \tag{15.70}$$

Similarly, we may verify that the net return of the bank at the time 0 is

$$f - 1 + \exp\left(-\int_0^s (X_t - K)^+\mathrm{d}t\right). \tag{15.71}$$

The fair price of this contract should make the borrower and the bank have an identical expected return, i.e.,

$$-f+1-E\left[\exp\left(-\int_0^s (X_t - K)^+\mathrm{d}t\right)\right] = f-1+E\left[\exp\left(-\int_0^s (X_t - K)^+\mathrm{d}t\right)\right].$$

Thus we have the following definition of the price of interest rate ceiling.

Definition 15.14 *(Zhang-Ralescu-Liu [201]) Assume an interest rate ceiling has a maximum interest rate K and a maturity date s. Then the price of the interest rate ceiling is*

$$f = 1 - E\left[\exp\left(-\int_0^s (X_t - K)^+\mathrm{d}t\right)\right]. \tag{15.72}$$

Theorem 15.14 *(Zhang-Ralescu-Liu [201]) Assume the uncertain interest rate X_t follows the uncertain differential equation (15.62). Then the price of the interest rate ceiling with a maximum interest rate K and a maturity date s is*

$$f = 1 - \int_0^1 \exp\left(-\int_0^s (X_t^\alpha - K)^+ \mathrm{d}t\right) \mathrm{d}\alpha \tag{15.73}$$

where X_t^α is the α-path of the corresponding uncertain differential equation.

Proof: It follows from Theorem 14.17 that the inverse uncertainty distribution of the time integral

$$\int_0^s (X_t - K)^+ \mathrm{d}t$$

is

$$\Psi_s^{-1}(\alpha) = \int_0^s (X_t^\alpha - K)^+ \mathrm{d}t.$$

Hence the price formula of the interest rate ceiling follows from Theorem 2.26 immediately.

Interest Rate Floor

An interest rate floor is a derivative contract in which the investor will not receive any less than a predetermined level of interest on his investment. Assume K is the minimum interest rate and s is the maturity date. For simplicity, we also assume the amount of investment is always 1 dollar.

Let f represent the price of this contract. Then the investor pays f for buying the contract at time 0, and has a payoff

$$\exp\left(\int_0^s X_t \vee K \mathrm{d}t\right) - \exp\left(\int_0^s X_t \mathrm{d}t\right) \tag{15.74}$$

at the maturity date s. Considering the time value of money, the present value of the payoff is

$$\begin{aligned}
&\exp\left(-\int_0^s X_t \mathrm{d}t\right)\left(\exp\left(\int_0^s X_t \vee K \mathrm{d}t\right) - \exp\left(\int_0^s X_t \mathrm{d}t\right)\right) \\
&= \exp\left(-\int_0^s X_t \mathrm{d}t + \int_0^s X_t \vee K \mathrm{d}t\right) - 1 \\
&= \exp\left(\int_0^s (K - X_t)^+ \mathrm{d}t\right) - 1.
\end{aligned}$$

Thus the net return of the investor at time 0 is

$$-f + \exp\left(\int_0^s (K - X_t)^+ \mathrm{d}t\right) - 1. \tag{15.75}$$

Similarly, we may verify that the net return of the bank at the time 0 is

$$f - \exp\left(\int_0^s (K - X_t)^+ \mathrm{d}t\right) + 1. \tag{15.76}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$-f + E\left[\exp\left(\int_0^s (K - X_t)^+ \mathrm{d}t\right)\right] - 1 = f - E\left[\exp\left(\int_0^s (K - X_t)^+ \mathrm{d}t\right)\right] + 1.$$

Thus we have the following definition of the price of interest rate floor.

Definition 15.15 *(Zhang-Ralescu-Liu [201]) Assume an interest rate floor has a minimum interest rate K and a maturity date s. Then the price of the interest rate floor is*

$$f = E\left[\exp\left(\int_0^s (K - X_t)^+ \mathrm{d}t\right)\right] - 1. \tag{15.77}$$

Theorem 15.15 *(Zhang-Ralescu-Liu [201]) Assume the uncertain interest rate X_t follows the uncertain differential equation (15.62). Then the price of the interest rate floor with a minimum interest rate K and a maturity date s is*

$$f = \int_0^1 \exp\left(\int_0^s (K - X_t^\alpha)^+ \mathrm{d}t\right) \mathrm{d}\alpha - 1 \tag{15.78}$$

where X_t^α is the α-path of the corresponding uncertain differential equation.

Proof: It follows from Theorem 14.18 that the inverse uncertainty distribution of the time integral

$$\int_0^s (K - X_t)^+ \mathrm{d}t$$

is

$$\Psi_s^{-1}(\alpha) = \int_0^s (K - X_t^{1-\alpha})^+ \mathrm{d}t.$$

Hence the price formula of the interest rate floor follows from Theorem 2.26 immediately.

15.8 Uncertain Currency Model

Liu-Chen-Ralescu [108] assumed that the exchange rate follows an uncertain differential equation and proposed an uncertain currency model,

$$\begin{cases} \mathrm{d}X_t = uX_t\mathrm{d}t & \text{(Domestic Currency)} \\ \mathrm{d}Y_t = vY_t\mathrm{d}t & \text{(Foreign Currency)} \\ \mathrm{d}Z_t = eZ_t\mathrm{d}t + \sigma Z_t\mathrm{d}C_t & \text{(Exchange Rate)} \end{cases} \tag{15.79}$$

where X_t represents the domestic currency with domestic interest rate u, Y_t represents the foreign currency with foreign interest rate v, and Z_t represents the exchange rate that is domestic currency price of one unit of foreign currency at time t. Note that the domestic currency price is $X_t = X_0 \exp(ut)$, the foreign currency price is $Y_t = Y_0 \exp(vt)$, and the exchange rate is

$$Z_t = Z_0 \exp(et + \sigma C_t) \tag{15.80}$$

whose inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = Z_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right). \tag{15.81}$$

European Currency Option

Definition 15.16 *A European currency option is a contract that gives the holder the right to exchange one unit of foreign currency at an expiration time s for K units of domestic currency.*

Suppose that the price of this contract is f in domestic currency. Then the investor pays f for buying the contract at time 0, and receives $(Z_s - K)^+$ in domestic currency at the expiration time s. Thus the net return of the investor at time 0 is

$$-f + \exp(-us)(Z_s - K)^+. \tag{15.82}$$

On the other hand, the bank receives f for selling the contract at time 0, and pays $(1 - K/Z_s)^+$ in foreign currency at the expiration time s. Thus the net return of the bank at the time 0 is

$$f - \exp(-vs)Z_0(1 - K/Z_s)^+. \tag{15.83}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$-f + \exp(-us)E[(Z_s - K)^+] = f - \exp(-vs)Z_0 E[(1 - K/Z_s)^+]. \tag{15.84}$$

Thus the European currency option price is given by the definition below.

Definition 15.17 *(Liu-Chen-Ralescu [108]) Assume a European currency option has a strike price K and an expiration time s. Then the European currency option price is*

$$f = \frac{1}{2}\exp(-us)E[(Z_s - K)^+] + \frac{1}{2}\exp(-vs)Z_0 E[(1 - K/Z_s)^+]. \tag{15.85}$$

Theorem 15.16 *(Liu-Chen-Ralescu [108]) Assume a European currency option for the uncertain currency model (15.79) has a strike price K and an expiration time s. Then the European currency option price is*

$$f = \frac{1}{2}\exp(-us)\int_0^1 \left(Z_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+ \mathrm{d}\alpha$$
$$+\frac{1}{2}\exp(-vs)\int_0^1 \left(Z_0 - K/\exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha.$$

Proof: Since $(Z_s - K)^+$ and $Z_0(1 - K/Z_s)^+$ are increasing functions with respect to Z_s, they have inverse uncertainty distributions

$$\Psi_s^{-1}(\alpha) = \left(Z_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+,$$

$$\Upsilon_s^{-1}(\alpha) = \left(Z_0 - K/\exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+,$$

respectively. Thus the European currency option price formula follows from Definition 15.17 immediately.

Remark 15.5: The European currency option price of the uncertain currency model (15.79) is a decreasing function of K, u and v.

Example 15.5: Assume the domestic interest rate $u = 0.08$, the foreign interest rate $v = 0.07$, the log-drift $e = 0.06$, the log-diffusion $\sigma = 0.32$, the initial exchange rate $Z_0 = 5$, the strike price $K = 6$ and the expiration time $s = 2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the European currency option price $f = 0.977$.

American Currency Option

Definition 15.18 *An American currency option is a contract that gives the holder the right to exchange one unit of foreign currency at any time prior to an expiration time s for K units of domestic currency.*

Suppose that the price of this contract is f in domestic currency. Then the investor pays f for buying the contract, and receives

$$\sup_{0\le t\le s} \exp(-ut)(Z_t - K)^+ \tag{15.86}$$

in domestic currency. Thus the net return of the investor at time 0 is

$$-f + \sup_{0\le t\le s} \exp(-ut)(Z_t - K)^+. \tag{15.87}$$

On the other hand, the bank receives f for selling the contract, and pays

$$\sup_{0\le t\le s} \exp(-vt)(1-K/Z_t)^+. \tag{15.88}$$

in foreign currency. Thus the net return of the bank at time 0 is

$$f - \sup_{0\le t\le s} \exp(-vt)Z_0(1-K/Z_t)^+. \tag{15.89}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$\begin{aligned} &-f + E\left[\sup_{0\le t\le s} \exp(-ut)(Z_t-K)^+\right] \\ &\quad = f - E\left[\sup_{0\le t\le s} \exp(-vt)Z_0(1-K/Z_t)^+\right]. \end{aligned} \tag{15.90}$$

Thus the American currency option price is given by the definition below.

Definition 15.19 *(Liu-Chen-Ralescu [108]) Assume an American currency option has a strike price K and an expiration time s. Then the American currency option price is*

$$f = \frac{1}{2}E\left[\sup_{0\le t\le s} \exp(-ut)(Z_t-K)^+\right] + \frac{1}{2}E\left[\sup_{0\le t\le s} \exp(-vt)Z_0(1-K/Z_t)^+\right].$$

Theorem 15.17 *(Liu-Chen-Ralescu [108]) Assume an American currency option for the uncertain currency model (15.79) has a strike price K and an expiration time s. Then the American currency option price is*

$$\begin{aligned} f = &\frac{1}{2}\int_0^1 \sup_{0\le t\le s} \exp(-ut)\left(Z_0\exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+ \mathrm{d}\alpha \\ &+\frac{1}{2}\int_0^1 \sup_{0\le t\le s} \exp(-vt)\left(Z_0 - K/\exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha. \end{aligned}$$

Proof: It follows from Theorem 14.13 that $\sup_{0\le t\le s}\exp(-ut)(Z_t-K)^+$ and $\sup_{0\le t\le s}\exp(-vt)Z_0(1-K/Z_t)^+$ have inverse uncertainty distributions

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-ut)\left(Z_0\exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+,$$

$$\Upsilon_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-vt)\left(Z_0 - K/\exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+,$$

respectively. Thus the American currency option price formula follows from Definition 15.19 immediately.

General Currency Model

If the exchange rate follows a general uncertain differential equation, then we have a general currency model,

$$\begin{cases} \mathrm{d}X_t = uX_t\mathrm{d}t \quad \text{(Domestic Currency)} \\ \mathrm{d}Y_t = vY_t\mathrm{d}t \quad \text{(Foreign Currency)} \\ \mathrm{d}Z_t = F(t, Z_t)\mathrm{d}t + G(t, Z_t)\mathrm{d}C_t \quad \text{(Exchange Rate)} \end{cases} \tag{15.91}$$

where u and v are interest rates, F and G are two functions, and C_t is a Liu process.

Theorem 15.18 *(Liu [94]) Assume a European currency option for the uncertain currency model (15.91) has a strike price K and an expiration time s. Then the European currency option price is*

$$f = \frac{1}{2}\int_0^1 \left(\exp(-us)(Z_s^\alpha - K)^+ + \exp(-vs)Z_0(1 - K/Z_s^\alpha)^+\right) \mathrm{d}\alpha \tag{15.92}$$

where Z_t^α is the α-path of the corresponding uncertain differential equation.

Proof: It follows from the fair price principle that the European option price is

$$f = \frac{1}{2}\exp(-us)E[(Z_s - K)^+] + \frac{1}{2}\exp(-vs)Z_0E[(1 - K/Z_s)^+]. \tag{15.93}$$

By using Theorem 14.12, we get the equation (15.92).

Theorem 15.19 *(Liu [94]) Assume an American currency option for the uncertain currency model (15.91) has a strike price K and an expiration time s. Then the American currency option price is*

$$f = \frac{1}{2}\int_0^1 \left(\sup_{0\le t\le s} \exp(-ut)(Z_t^\alpha - K)^+ + \sup_{0\le t\le s} \exp(-vt)Z_0(1 - K/Z_t^\alpha)^+\right) \mathrm{d}\alpha$$

where Z_t^α is the α-path of the corresponding uncertain differential equation.

Proof: It follows from the fair price principle that the American option price is

$$f = \frac{1}{2}E\left[\sup_{0\le t\le s} \exp(-ut)(Z_t - K)^+\right] + \frac{1}{2}E\left[\sup_{0\le t\le s} \exp(-vt)Z_0(1 - K/Z_t)^+\right].$$

By using Theorem 14.13, we get the result.

15.9 Bibliographic Notes

The classical finance theory assumed that stock price, interest rate, and exchange rate follow stochastic differential equations. However, this preassumption was challenged among others by Liu [88] in which a convincing paradox was presented to show why the real stock price is impossible to follow any stochastic differential equations (see also Appendix B.9). As an alternative, Liu [88] suggested to develop a theory of uncertain finance.

Uncertain differential equations were first introduced into finance by Liu [79] in 2009 in which an uncertain stock model was proposed and European option price formulas were provided. Besides, Chen [6] derived American option price formulas, Sun-Chen [142] and Zhang-Liu [200] verified Asian option price formulas, and Yao [173] proved a no-arbitrage theorem for this type of uncertain stock model. It is emphasized that uncertain stock models were also actively investigated among others by Peng-Yao [118], Yu [188], Chen-Liu-Ralescu [12], Yao [178], and Ji-Zhou [61].

Uncertain differential equations were used to simulate floating interest rate by Chen-Gao [14] in 2013. Following that, Jiao-Yao [63] presented a price formula of zero-coupon bond, and Zhang-Ralescu-Liu [201] discussed the valuation of interest rate ceiling and floor.

Uncertain differential equations were employed to model currency exchange rate by Liu-Chen-Ralescu [108] in 2015 in which some currency option price formulas were derived for the uncertain currency markets. Afterwards, uncertain currency models were also actively investigated among others by Liu [94], Shen-Yao [134] and Wang-Ning [148].

For further explorations on the development of the theory of uncertain finance, the interested reader may consult Chen's book [17].

Chapter 16

Uncertain Statistics

The study of uncertain statistics was started by Liu [83] in 2010. It is a methodology for collecting and interpreting expert's experimental data by uncertainty theory. This chapter will design a questionnaire survey for collecting expert's experimental data, and introduce linear interpolation method, principle of least squares, method of moments, and Delphi method for determining uncertainty distributions and membership functions from the expert's experimental data. In addition, uncertain regression analysis and uncertain time series analysis are also documented in this chapter.

16.1 Expert's Experimental Data

Uncertain statistics is based on expert's experimental data rather than historical data. How do we obtain expert's experimental data? Liu [83] proposed a questionnaire survey for collecting expert's experimental data. The starting point is to invite one or more domain experts who are asked to complete a questionnaire about the meaning of an uncertain variable ξ like "how far from Beijing to Tianjin".

We first ask the domain expert to choose a possible value x (say 110km) that the uncertain variable ξ may take, and then quiz him

$$\text{“How likely is } \xi \text{ less than or equal to } x\text{?”} \tag{16.1}$$

Denote the expert's belief degree by α (say 0.6). Note that the expert's belief degree of ξ greater than x must be $1-\alpha$ due to the self-duality of uncertain measure. An expert's experimental data

$$(x, \alpha) = (110, 0.6) \tag{16.2}$$

is thus acquired from the domain expert.

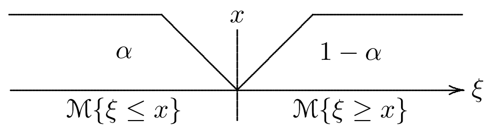


Figure 16.1: Expert's Experimental Data (x, α)

Repeating the above process, the following expert's experimental data are obtained by the questionnaire,

$$(x_1, \alpha_1),\ (x_2, \alpha_2),\ \cdots,\ (x_n, \alpha_n). \tag{16.3}$$

Remark 16.1: None of x, α and n could be assigned a value in the questionnaire before asking the domain expert. Otherwise, the domain expert may have no knowledge or experiments enough to answer your questions.

16.2 Questionnaire Survey

Beijing is the capital of China, and Tianjin is a coastal city. Assume that the real distance between them is not exactly known for us, and is regarded as an uncertain variable. Chen-Ralescu [11] employed uncertain statistics to estimate the travel distance between Beijing and Tianjin. The consultation process is as follows:

Q1: May I ask you how far is from Beijing to Tianjin? What do you think is the minimum distance?

A1: 100km. *(an expert's experimental data* $(100, 0)$ *is acquired)*

Q2: What do you think is the maximum distance?

A2: 150km. *(an expert's experimental data* $(150, 1)$ *is acquired)*

Q3: What do you think is a likely distance?

A3: 130km.

Q4: To what degree do you think that the real distance is less than 130km?

A4: 60%. *(an expert's experimental data* $(130, 0.6)$ *is acquired)*

Q5: Is there another number this distance may be? If yes, what is it?

A5: 140km.

Q6: To what degree do you think that the real distance is less than 140km?

A6: 90%. *(an expert's experimental data* $(140, 0.9)$ *is acquired)*

Q7: Is there another number this distance may be? If yes, what is it?

A7: 120km.

Q8: To what degree do you think that the real distance is less than 120km?

A8: 30%. *(an expert's experimental data* $(120, 0.3)$ *is acquired)*

Q9: Is there another number this distance may be? If yes, what is it?

A9: No idea.

By using the questionnaire survey, five expert's experimental data of the travel distance between Beijing and Tianjin are acquired from the domain expert,

$$(100, 0),\ (120, 0.3),\ (130, 0.6),\ (140, 0.9),\ (150, 1). \tag{16.4}$$

16.3 Determining Uncertainty Distribution

In order to determine the uncertainty distribution of uncertain variable, this section will introduce empirical uncertainty distribution (i.e., linear interpolation method), principle of least squares, method of moments, and Delphi method.

Empirical Uncertainty Distribution

How do we determine the uncertainty distribution for an uncertain variable? Assume that we have obtained a set of expert's experimental data

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n) \tag{16.5}$$

that meet the following consistence condition (perhaps after a rearrangement)

$$x_1 < x_2 < \cdots < x_n, \quad 0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1. \tag{16.6}$$

Based on those expert's experimental data, Liu [83] suggested an empirical uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x < x_1 \\ \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \le x \le x_{i+1},\ 1 \le i < n \\ 1, & \text{if } x > x_n. \end{cases} \tag{16.7}$$

Essentially, it is a type of linear interpolation method.

The empirical uncertainty distribution Φ determined by (16.7) has an expected value

$$E[\xi] = \frac{\alpha_1 + \alpha_2}{2} x_1 + \sum_{i=2}^{n-1} \frac{\alpha_{i+1} - \alpha_{i-1}}{2} x_i + \left(1 - \frac{\alpha_{n-1} + \alpha_n}{2}\right) x_n. \tag{16.8}$$

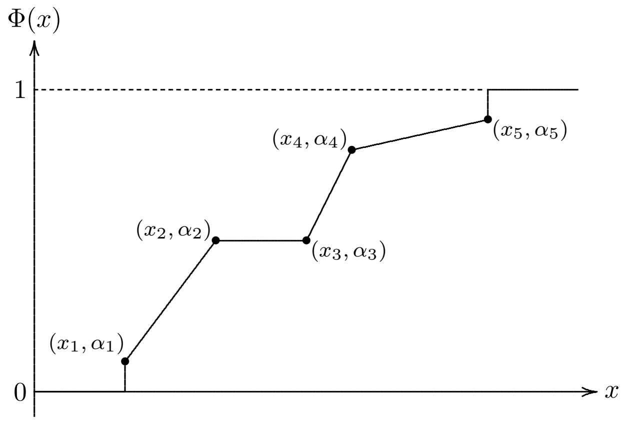


Figure 16.2: Empirical Uncertainty Distribution $\Phi(x)$

If all x_i's are nonnegative, then the k-th empirical moments are

$$E[\xi^k] = \alpha_1 x_1^k + \frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}(\alpha_{i+1}-\alpha_i)x_i^j x_{i+1}^{k-j} + (1-\alpha_n)x_n^k. \tag{16.9}$$

Example 16.1: Recall that the five expert's experimental data $(100,0)$, $(120,0.3)$, $(130,0.6)$, $(140,0.9)$, $(150,1)$ of the travel distance between Beijing and Tianjin have been acquired in Section 16.2. Based on those expert's experimental data, an empirical uncertainty distribution of travel distance is shown in Figure 16.3.

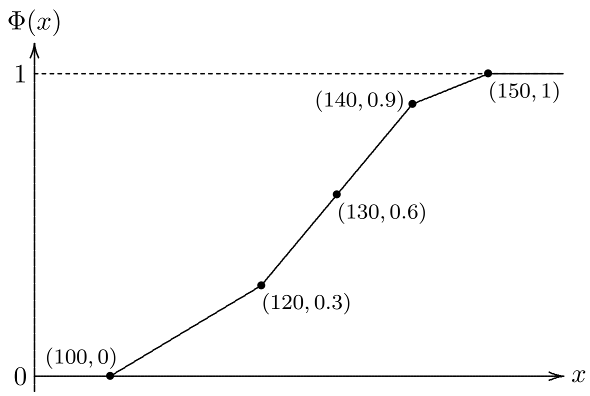


Figure 16.3: Empirical Uncertainty Distribution of Travel Distance between Beijing and Tianjin. Note that the empirical expected distance is 125.5km and the real distance is 127km in the google earth.

Principle of Least Squares

Assume that an uncertainty distribution to be determined has a known functional form $\Phi(x|\theta)$ with an unknown parameter θ. In order to estimate the parameter θ, Liu [83] employed the principle of least squares that minimizes the sum of the squares of the distance of the expert's experimental data to the uncertainty distribution. This minimization can be performed in either the vertical or horizontal direction. If the expert's experimental data

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n) \tag{16.10}$$

are obtained and the vertical direction is accepted, then we have

$$\min_\theta \sum_{i=1}^n (\Phi(x_i|\theta) - \alpha_i)^2. \tag{16.11}$$

The optimal solution $\hat{\theta}$ of (16.11) is called the least squares estimate of θ, and then the least squares uncertainty distribution is $\Phi(x|\hat{\theta})$.

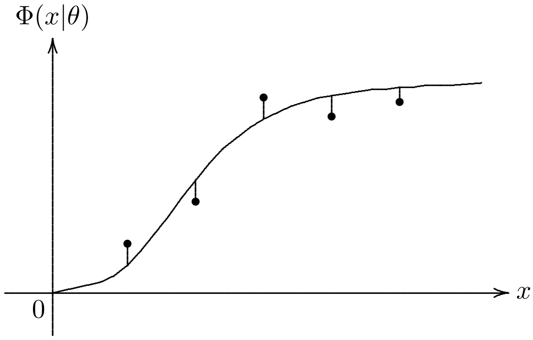


Figure 16.4: Principle of Least Squares

Example 16.2: Assume that an uncertainty distribution has a linear form with two unknown parameters a and b, i.e.,

$$\Phi(x|a,b) = \begin{cases} 0, & \text{if } x \le a \\ \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x \ge b. \end{cases} \tag{16.12}$$

We also assume the following expert's experimental data,

$$(1, 0.15), (2, 0.45), (3, 0.55), (4, 0.85), (5, 0.95). \tag{16.13}$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield that $\hat{a} = 0.2273$, $\hat{b} = 4.7727$ and the least squares uncertainty distribution is

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 0.2273 \\ (x - 0.2273)/4.5454, & \text{if } 0.2273 \le x \le 4.7727 \\ 1, & \text{if } x \ge 4.7727. \end{cases} \tag{16.14}$$

Example 16.3: Assume that an uncertainty distribution has a lognormal form with two unknown parameters e and σ, i.e.,

$$\Phi(x|e, \sigma) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}. \tag{16.15}$$

We also assume the following expert's experimental data,

$$(0.6, 0.1),\ (1.0, 0.3),\ (1.5, 0.4),\ (2.0, 0.6),\ (2.8, 0.8),\ (3.6, 0.9). \tag{16.16}$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield that $\hat{e} = 0.4825$, $\hat{\sigma} = 0.7852$ and the least squares uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{0.4825 - \ln x}{0.4329}\right)\right)^{-1}. \tag{16.17}$$

Method of Moments

Assume that a nonnegative uncertain variable has an uncertainty distribution

$$\Phi(x|\theta_1, \theta_2, \cdots, \theta_p) \tag{16.18}$$

with unknown parameters $\theta_1, \theta_2, \cdots, \theta_p$. Given a set of expert's experimental data

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n) \tag{16.19}$$

with

$$0 \le x_1 < x_2 < \cdots < x_n, \quad 0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1, \tag{16.20}$$

Wang-Peng [152] proposed a method of moments to estimate the unknown parameters of uncertainty distribution. At first, the kth empirical moments of the expert's experimental data are defined as that of the corresponding empirical uncertainty distribution, i.e.,

$$\overline{\xi^k} = \alpha_1 x_1^k + \frac{1}{k+1} \sum_{i=1}^{n-1} \sum_{j=0}^{k} (\alpha_{i+1} - \alpha_i) x_i^j x_{i+1}^{k-j} + (1 - \alpha_n) x_n^k. \tag{16.21}$$

The moment estimates $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_p$ are then obtained by equating the first p moments of $\Phi(x|\theta_1, \theta_2, \cdots, \theta_p)$ to the corresponding first p empirical moments. In other words, the moment estimates $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_p$ should solve the system of equations,

$$\int_0^{+\infty} (1 - \Phi(\sqrt[k]{x} \,|\, \theta_1, \theta_2, \cdots, \theta_p))\mathrm{d}x = \overline{\xi^k}, \quad k = 1, 2, \cdots, p \tag{16.22}$$

where $\overline{\xi^1}, \overline{\xi^2}, \cdots, \overline{\xi^p}$ are empirical moments determined by (16.21).

Example 16.4: Assume that a questionnaire survey has successfully acquired the following expert's experimental data,

$$(1.2, 0.1),\ (1.5, 0.3),\ (1.8, 0.4),\ (2.5, 0.6),\ (3.9, 0.8),\ (4.6, 0.9). \tag{16.23}$$

Then the first three empirical moments are 2.5100, 7.7226 and 29.4936. We also assume that the uncertainty distribution to be determined has a zigzag form with three unknown parameters a, b and c, i.e.,

$$\Phi(x|a, b, c) = \begin{cases} 0, & \text{if } x \le a \\ \dfrac{x - a}{2(b - a)}, & \text{if } a \le x \le b \\ \dfrac{x + c - 2b}{2(c - b}, & \text{if } b \le x \le c \\ 1, & \text{if } x \ge c. \end{cases} \tag{16.24}$$

From the expert's experimental data, we may believe that the unknown parameters must be positive numbers. Thus the first three moments of the zigzag uncertainty distribution $\Phi(x|a, b, c)$ are

$$\frac{a + 2b + c}{4},$$

$$\frac{a^2 + ab + 2b^2 + bc + c^2}{6},$$

$$\frac{a^3 + a^2b + ab^2 + 2b^3 + b^2c + bc^2 + c^3}{8}.$$

It follows from the method of moments that the unknown parameters a, b, c should solve the system of equations,

$$\begin{cases} a + 2b + c = 4 \times 2.5100 \\ a^2 + ab + 2b^2 + bc + c^2 = 6 \times 7.7226 \\ a^3 + a^2b + ab^2 + 2b^3 + b^2c + bc^2 + c^3 = 8 \times 29.4936. \end{cases} \tag{16.25}$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield that the moment estimates are $(\hat{a}, \hat{b}, \hat{c}) = (0.9804, 2.0303, 4.9991)$ and the corresponding uncertainty distribution is

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 0.9804 \\ (x - 0.9804)/2.0998, & \text{if } 0.9804 \le x \le 2.0303 \\ (x + 0.9385)/5.9376, & \text{if } 2.0303 \le x \le 4.9991 \\ 1, & \text{if } x \ge 4.9991. \end{cases} \tag{16.26}$$

Multiple Domain Experts

Assume there are m domain experts and each produces an uncertainty distribution. Then we may get m uncertainty distributions $\Phi_1(x), \Phi_2(x), \cdots, \Phi_m(x)$. It was suggested by Liu [83] that the m uncertainty distributions should be aggregated to an uncertainty distribution

$$\Phi(x) = w_1\Phi_1(x) + w_2\Phi_2(x) + \cdots + w_m\Phi_m(x) \tag{16.27}$$

where $w_1, w_2, \cdots, w_m$ are convex combination coefficients (i.e., they are nonnegative numbers and $w_1 + w_2 + \cdots + w_n = 1$) representing weights of the domain experts. For example, we may set

$$w_i = \frac{1}{m}, \quad \forall i = 1, 2, \cdots, n. \tag{16.28}$$

Since $\Phi_1(x), \Phi_2(x), \cdots, \Phi_m(x)$ are uncertainty distributions, they are increasing functions taking values in $[0, 1]$ and are not identical to either 0 or 1. It is easy to verify that their convex combination $\Phi(x)$ is also an increasing function taking values in $[0, 1]$ and $\Phi(x) \not\equiv 0$, $\Phi(x) \not\equiv 1$. Hence $\Phi(x)$ is also an uncertainty distribution by Peng-Iwamura theorem.

Delphi Method

Delphi method was originally developed in the 1950s by the RAND Corporation based on the assumption that group experience is more valid than individual experience. This method asks the domain experts answer questionnaires in two or more rounds. After each round, a facilitator provides an anonymous summary of the answers from the previous round as well as the reasons that the domain experts provided for their opinions. Then the domain experts are encouraged to revise their earlier answers in light of the summary. It is believed that during this process the opinions of domain experts will converge to an appropriate answer. Wang-Gao-Guo [150] recast Delphi method as a process to determine uncertainty distributions. The main steps are listed as follows:

Step 1. The m domain experts provide their expert's experimental data,

$$(x_{ij}, \alpha_{ij}), \quad j = 1, 2, \cdots, n_i,\ i = 1, 2, \cdots, m. \tag{16.29}$$

Step 2. Use the i-th expert's experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})$ to generate the uncertainty distributions Φ_i of the i-th domain experts, $i = 1, 2, \cdots, m$, respectively.

Step 3. Compute $\Phi(x) = w_1\Phi_1(x) + w_2\Phi_2(x) + \cdots + w_m\Phi_m(x)$ where $w_1, w_2, \cdots, w_m$ are convex combination coefficients representing weights of the domain experts.

Step 4. If $|\alpha_{ij} - \Phi(x_{ij})|$ are less than a given level $\varepsilon > 0$ for all i and j, then go to Step 5. Otherwise, the i-th domain experts receive the summary (for example, the function Φ obtained in the previous round and the reasons of other experts), and then provide a set of revised expert's experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})$ for $i = 1, 2, \cdots, m$. Go to Step 2.

Step 5. The last function Φ is the uncertainty distribution to be determined.

16.4 Determining Membership Function

In order to determine the membership function of uncertain set, this section will introduce empirical membership function (i.e., linear interpolation method) and principle of least squares.

Expert's Experimental Data

Expert's experimental data were suggested by Liu [84] to represent expert's knowledge about the membership function to be determined. The first step is to ask the domain expert to choose a possible point x that the uncertain set ξ may contain, and then quiz him

$$\text{"How likely does } x \text{ belong to } \xi\text{?"} \tag{16.30}$$

Assume the expert's belief degree is α in uncertain measure. Note that the expert's belief degree of x not belonging to ξ must be $1-\alpha$ due to the duality of uncertain measure. An expert's experimental data (x, α) is thus acquired from the domain expert. Repeating the above process, the following expert's experimental data are obtained by the questionnaire,

$$(x_1, \alpha_1),\ (x_2, \alpha_2),\ \cdots,\ (x_n, \alpha_n). \tag{16.31}$$

Empirical Membership Function

How do we determine the membership function for an uncertain set? The first method is the linear interpolation method developed by Liu [84]. Assume that we have obtained a set of expert's experimental data

$$(x_1, \alpha_1),\ (x_2, \alpha_2),\ \cdots,\ (x_n, \alpha_n). \tag{16.32}$$

Without loss of generality, we also assume $x_1 < x_2 < \cdots < x_n$. Based on those expert's experimental data, an empirical membership function is determined as follows,

$$\mu(x) = \begin{cases} \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \le x \le x_{i+1},\ 1 \le i < n \\ 0, & \text{otherwise.} \end{cases}$$

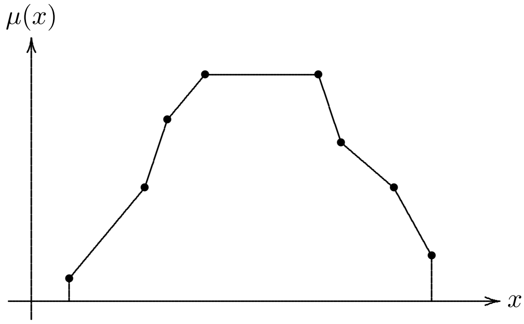


Figure 16.5: Empirical Membership Function $\mu(x)$

Principle of Least Squares

Principle of least squares was first employed to determine membership function by Liu [84]. Assume that a membership function to be determined has a known functional form $\mu(x|\theta)$ with an unknown parameter θ. In order to estimate the parameter θ, we may employ the principle of least squares that minimizes the sum of the squares of the distance of the expert's experimental data to the membership function. If the expert's experimental data

$$(x_1, \alpha_1),\ (x_2, \alpha_2),\ \cdots,\ (x_n, \alpha_n) \tag{16.33}$$

are obtained, then we have

$$\min_\theta \sum_{i=1}^n (\mu(x_i|\theta) - \alpha_i)^2. \tag{16.34}$$

The optimal solution $\hat\theta$ of (16.34) is called the least squares estimate of θ, and then the least squares membership function is $\mu(x|\hat\theta)$.

Example 16.5: Assume that a membership function has a trapezoidal form (a, b, c, d). We also assume the following expert's experimental data,

$$(1, 0.15),\ (2, 0.45),\ (3, 0.90),\ (6, 0.85),\ (7, 0.60),\ (8, 0.20). \tag{16.35}$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield that the least squares membership function has a trapezoidal form $(0.6667, 3.3333, 5.6154, 8.6923)$.

What is "about 100km"?

Let us pay attention to the concept of "about 100km". When we are interested in what distances can be considered "about 100km", it is reasonable to regard such a concept as an uncertain set. In order to determine the membership function of "about 100km", a questionnaire survey was made for collecting expert's experimental data. The consultation process is as follows:

Q1: May I ask you what distances belong to "about 100km"? What do you think is the minimum distance?

A1: 80km. *(an expert's experimental data* $(80, 0)$ *is acquired)*

Q2: What do you think is the maximum distance?

A2: 120km. *(an expert's experimental data* $(120, 0)$ *is acquired)*

Q3: What distance do you think belongs to "about 100km"?

A3: 95km.

Q4: To what degree do you think that 95km belongs to "about 100km"?

A4: 100%. *(an expert's experimental data* $(95, 1)$ *is acquired)*

Q5: Is there another distance that belongs to "about 100km"? If yes, what is it?

A5: 105km.

Q6: To what degree do you think that 105km belongs to "about 100km"?

A6: 100%. *(an expert's experimental data* $(105, 1)$ *is acquired)*

Q7: Is there another distance that belongs to "about 100km"? If yes, what is it?

A7: 90km.

Q8: To what degree do you think that 90km belongs to "about 100km"?

A8: 50%. *(an expert's experimental data* $(90, 0.5)$ *is acquired)*

Q9: Is there another distance that belongs to "about 100km"? If yes, what is it?

A9: 110km.

Q10: To what degree do you think that 110km belongs to "about 100km"?

A10: 50%. *(an expert's experimental data* $(110, 0.5)$ *is acquired)*

Q11: Is there another distance that belongs to "about 100km"? If yes, what is it?

A11: No idea.

Until now six expert's experimental data $(80, 0)$, $(90, 0.5)$, $(95, 1)$, $(105, 1)$, $(110, 0.5)$, $(120, 0)$ are acquired from the domain expert. Based on those expert's experimental data, an empirical membership function of "about 100km" is produced and shown by Figure 16.6.

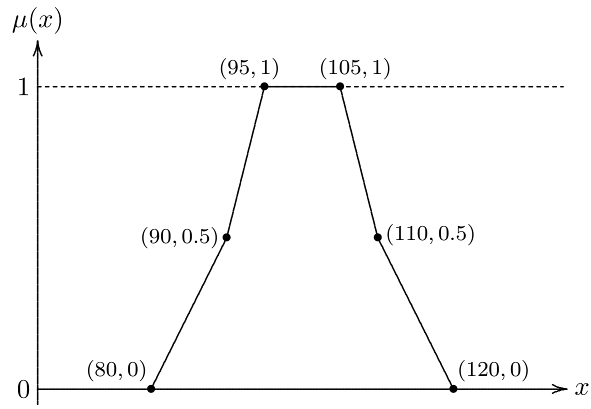


Figure 16.6: Empirical Membership Function of "about 100km"

16.5 Uncertain Regression Analysis

Let $(x_1, x_2, \cdots, x_p)$ be a vector of explanatory variables, and let y be a response variable. Assume the functional relationship between $(x_1, x_2, \cdots, x_p)$ and y is expressed by a regression model

$$y = f(x_1, x_2, \cdots, x_p|\boldsymbol{\beta}) + \varepsilon \tag{16.36}$$

where $\boldsymbol{\beta}$ is an unknown vector of parameters, and ε is a disturbance term. Especially, we will call

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon \tag{16.37}$$

a linear regression model, and call

$$y = \beta_0 - \beta_1 \exp(-\beta_2 x) + \varepsilon, \quad \beta_1 > 0, \beta_2 > 0 \tag{16.38}$$

an asymptotic regression model.

Traditionally, it is assumed that $(x_1, x_2, \cdots, x_p, y)$ are able to be precisely observed. However, in many cases, the observations of those data are imprecise and characterized in terms of uncertain variables. It is thus assumed that we have a set of imprecisely observed data,

$$(\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i), \quad i = 1, 2, \cdots, n \tag{16.39}$$

where $\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i$ are uncertain variables with uncertainty distributions $\Phi_{i1}, \Phi_{i2}, \cdots, \Phi_{ip}, \Psi_i$, $i = 1, 2, \cdots, n$, respectively.

Based on the imprecisely observed data (16.39), Yao-Liu [184] suggested that the least squares estimate of $\boldsymbol{\beta}$ in the regression model

$$y = f(x_1, x_2, \cdots, x_p|\boldsymbol{\beta}) + \varepsilon \tag{16.40}$$

is the solution of the minimization problem,

$$\min_{\boldsymbol{\beta}} \sum_{i=1}^{n} E[(\tilde{y}_i - f(\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}|\boldsymbol{\beta}))^2]. \tag{16.41}$$

If the minimization solution is $\boldsymbol{\beta}^*$, then the fitted regression model is determined by

$$y = f(x_1, x_2, \cdots, x_p|\boldsymbol{\beta}^*). \tag{16.42}$$

Theorem 16.1 *Let $(\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i)$, $i = 1, 2, \cdots, n$ be a set of imprecisely observed data, where $\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i$ are independent uncertain variables with regular uncertainty distributions $\Phi_{i1}, \Phi_{i2}, \cdots, \Phi_{ip}, \Psi_i$, $i = 1, 2, \cdots, n$, respectively. Then the least squares estimate of $\beta_0, \beta_1, \cdots, \beta_p$ in the linear regression model*

$$y = \beta_0 + \sum_{j=1}^{p} \beta_j x_j + \varepsilon \tag{16.43}$$

solves the minimization problem,

$$\min_{\beta_0, \beta_1, \cdots, \beta_p} \sum_{i=1}^{n} \int_0^1 \left(\Psi_i^{-1}(\alpha) - \beta_0 - \sum_{j=1}^{p} \beta_j \Upsilon_{ij}^{-1}(\alpha) \right)^2 \mathrm{d}\alpha \tag{16.44}$$

where

$$\Upsilon_{ij}^{-1}(\alpha) = \begin{cases} \Phi_{ij}^{-1}(1-\alpha), & \text{if } \beta_j \ge 0 \\ \Phi_{ij}^{-1}(\alpha), & \text{if } \beta_j < 0 \end{cases} \tag{16.45}$$

for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, p$.

Proof: Note that the least squares estimate of $\beta_0, \beta_1, \cdots, \beta_p$ in the linear regression model is the solution of the minimization problem,

$$\min_{\beta_0,\beta_1,\cdots,\beta_p} \sum_{i=1}^{n} E\left[\left(\tilde{y}_i - \beta_0 - \sum_{j=1}^{p} \beta_j \tilde{x}_{ij}\right)^2\right]. \tag{16.46}$$

For each index i, the inverse uncertainty distribution of the uncertain variable

$$\tilde{y}_i - \beta_0 - \sum_{j=1}^{p} \beta_j \tilde{x}_{ij}$$

is just

$$F_i^{-1}(\alpha) = \Psi_i^{-1}(\alpha) - \beta_0 - \sum_{j=1}^{p} \beta_j \Upsilon_{ij}^{-1}(\alpha).$$

It follows from Theorem 2.42 that

$$E\left[\left(\tilde{y}_i - \beta_0 - \sum_{j=1}^{p} \beta_j \tilde{x}_{ij}\right)^2\right] = \int_0^1 \left(\Psi_i^{-1}(\alpha) - \beta_0 - \sum_{j=1}^{p} \beta_j \Upsilon_{ij}^{-1}(\alpha)\right)^2 \mathrm{d}\alpha.$$

Hence the minimization problem (16.44) is equivalent to (16.46). The theorem is thus proved.

Exercise 16.1: Let $(\tilde{x}_i, \tilde{y}_i)$, $i = 1, 2, \cdots, n$ be a set of imprecisely observed data, where $\tilde{x}_i$ and $\tilde{y}_i$ are independent uncertain variables with regular uncertainty distributions Φ_i and Ψ_i, $i = 1, 2, \cdots, n$, respectively. Show that the least squares estimate of $\beta_0, \beta_1, \beta_2$ in the asymptotic regression model

$$y = \beta_0 - \beta_1 \exp(-\beta_2 x) + \varepsilon, \quad \beta_1 > 0, \beta_2 > 0 \tag{16.47}$$

solves the minimization problem,

$$\min_{\beta_0,\beta_1>0,\beta_2>0} \sum_{i=1}^{n} \int_0^1 \left(\Psi_i^{-1}(\alpha) - \beta_0 + \beta_1 \exp(-\beta_2 \Phi_i^{-1}(1-\alpha))\right)^2 \mathrm{d}\alpha. \tag{16.48}$$

Residual Error

Definition 16.1 *(Lio-Liu [73]) Let $(\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i)$, $i = 1, 2, \cdots, n$ be a set of imprecisely observed data, and let the fitted regression model be*

$$y = f(x_1, x_2, \cdots, x_p | \boldsymbol{\beta}^*). \tag{16.49}$$

Then for each index i ($i = 1, 2, \cdots, n$), the term

$$\hat{\varepsilon}_i = \tilde{y}_i - f(\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip} | \boldsymbol{\beta}^*) \tag{16.50}$$

is called the i-th residual error.

If the disturbance term ε is assumed to be an uncertain variable, then its expected value can be estimated as the average of the expected values of residual errors, i.e.,

$$\hat{e} = \frac{1}{n}\sum_{i=1}^{n} E[\hat{\varepsilon}_i] \tag{16.51}$$

and the variance can be estimated as

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} E[(\hat{\varepsilon}_i - \hat{e})^2] \tag{16.52}$$

where $\hat{\varepsilon}_i$ are the i-th residual errors, $i = 1, 2, \cdots, n$, respectively.

Theorem 16.2 *(Lio-Liu [73]) Let $(\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i)$, $i = 1, 2, \cdots, n$ be a set of imprecisely observed data, where $\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i$ are independent uncertain variables with regular uncertainty distributions $\Phi_{i1}, \Phi_{i2}, \cdots, \Phi_{ip}, \Psi_i$, $i = 1, 2, \cdots, n$, respectively, and let the fitted linear regression model be*

$$y = \beta_0^* + \sum_{j=1}^{p} \beta_j^* x_j. \tag{16.53}$$

Then the estimated expected value of disturbance term ε is

$$\hat{e} = \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \left(\Psi_i^{-1}(\alpha) - \beta_0^* - \sum_{j=1}^{p}\beta_j^*\Upsilon_{ij}^{-1}(\alpha)\right)\mathrm{d}\alpha \tag{16.54}$$

and the estimated variance is

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \left(\Psi_i^{-1}(\alpha) - \beta_0^* - \sum_{j=1}^{p}\beta_j^*\Upsilon_{ij}^{-1}(\alpha) - \hat{e}\right)^2\mathrm{d}\alpha \tag{16.55}$$

where

$$\Upsilon_{ij}^{-1}(\alpha) = \begin{cases} \Phi_{ij}^{-1}(1-\alpha), & \text{if } \beta_j^* \ge 0 \\ \Phi_{ij}^{-1}(\alpha), & \text{if } \beta_j^* < 0 \end{cases} \tag{16.56}$$

for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, p$.

Proof: For each index i, the inverse uncertainty distribution of the uncertain variable

$$\tilde{y}_i - \beta_0^* - \sum_{j=1}^{p}\beta_j^*\tilde{x}_{ij}$$

is just

$$F_i^{-1}(\alpha) = \Psi_i^{-1}(\alpha) - \beta_0^* - \sum_{j=1}^{p}\beta_j^*\Upsilon_{ij}^{-1}(\alpha).$$

It follows from Theorems 2.25 and 2.42 that (16.54) and (16.55) hold.

Exercise 16.2: Let $(\tilde{x}_i, \tilde{y}_i)$, $i = 1, 2, \cdots, n$ be a set of imprecisely observed data, where $\tilde{x}_i$ and $\tilde{y}_i$ are independent uncertain variables with regular uncertainty distributions Φ_i and Ψ_i, $i = 1, 2, \cdots, n$, respectively, and let the fitted asymptotic regression model be

$$y = \beta_0^* - \beta_1^* \exp(-\beta_2^* x), \quad \beta_1^* > 0, \beta_2^* > 0. \tag{16.57}$$

Show that the estimated expected value of disturbance term ε is

$$\hat{e} = \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \left(\Psi_i^{-1}(\alpha) - \beta_0^* + \beta_1^* \exp(-\beta_2^* \Phi_i^{-1}(1-\alpha))\right) \mathrm{d}\alpha \tag{16.58}$$

and the estimated variance is

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \left(\Psi_i^{-1}(\alpha) - \beta_0^* + \beta_1^* \exp(-\beta_2^* \Phi_i^{-1}(1-\alpha)) - \hat{e}\right)^2 \mathrm{d}\alpha. \tag{16.59}$$

Forecast Value and Confidence Interval

Now let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p)$ be a new explanatory vector, where $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_p$, respectively. Assume (i) the fitted linear regression model is

$$y = \beta_0^* + \sum_{j=1}^{p} \beta_j^* x_j, \tag{16.60}$$

and (ii) the disturbance term ε has expected value $\hat{e}$ and variance $\hat{\sigma}^2$, and is independent of $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p$. Lio-Liu [73] suggested that the *forecast uncertain variable* of response variable y with respect to $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p$ is determined by

$$\hat{y} = \beta_0^* + \sum_{j=1}^{p} \beta_j^* \tilde{x}_j + \varepsilon, \tag{16.61}$$

and the *forecast value* is defined as the expected value of the forecast uncertain variable $\hat{y}$, i.e.,

$$\mu = \beta_0^* + \sum_{j=1}^{p} \beta_j^* E[\tilde{x}_j] + \hat{e}. \tag{16.62}$$

If we suppose further that the disturbance term ε follows normal uncertainty distribution, then the inverse uncertainty distribution of forecast uncertain variable $\hat{y}$ is

$$\hat{\Psi}^{-1}(\alpha) = \beta_0^* + \sum_{j=1}^{p} \beta_j^* \Upsilon_j^{-1}(\alpha) + \Phi^{-1}(\alpha) \tag{16.63}$$

where

$$\Upsilon_j^{-1}(\alpha) = \begin{cases} \Phi_j^{-1}(\alpha), & \text{if } \beta_j^* \geq 0 \\ \Phi_j^{-1}(1-\alpha), & \text{if } \beta_j^* < 0 \end{cases} \tag{16.64}$$

for $j = 1, 2, \cdots, p$, and $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of $\mathcal{N}(\hat{e}, \hat{\sigma})$, i.e.,

$$\Phi^{-1}(\alpha) = \hat{e} + \frac{\hat{\sigma}\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{16.65}$$

From $\hat{\Psi}^{-1}$, we may also derive the uncertainty distribution $\hat{\Psi}$ of $\hat{y}$. Take α (e.g., 95%) as the confidence level, and find the minimum value b such that

$$\hat{\Psi}(\mu + b) - \hat{\Psi}(\mu - b) \geq \alpha. \tag{16.66}$$

Since $\mathcal{M}\{\mu - b \leq \hat{y} \leq \mu + b\} \geq \hat{\Psi}(\mu + b) - \hat{\Psi}(\mu - b) \geq \alpha$, Lio-Liu [73] suggested that the α *confidence interval* of response variable y is

$$\mu \pm b. \tag{16.67}$$

Exercise 16.3: Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p)$ be a new explanatory vector, where $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_p$, respectively. Assume (i) the fitted linear regression model is

$$y = \beta_0^* + \sum_{j=1}^{p} \beta_j^* x_j, \tag{16.68}$$

and (ii) the disturbance term ε follows linear uncertainty distribution with expected value $\hat{e}$ and variance $\hat{\sigma}^2$, and is independent of $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p$. What is the α confidence interval of response variable y? (Hint: The linear uncertain variable $\mathcal{L}(\hat{e} - \sqrt{3}\hat{\sigma}, \hat{e} + \sqrt{3}\hat{\sigma})$ has expected value $\hat{e}$ and variance $\hat{\sigma}^2$.)

Exercise 16.4: Let $\tilde{x}$ be a new explanatory variable with regular uncertainty distribution Φ. Assume (i) the fitted asymptotic regression model is

$$y = \beta_0^* - \beta_1^* \exp(-\beta_2^* x), \quad \beta_1^* > 0, \beta_2^* > 0, \tag{16.69}$$

and (ii) the disturbance term ε follows normal uncertainty distribution with expected value $\hat{e}$ and variance $\hat{\sigma}^2$, and is independent of $\tilde{x}$. What are the forecast value and α confidence interval of response variable y?

Example 16.6: Suppose that there exist 24 imprecisely observed data $(\tilde{x}_{i1}, \tilde{x}_{i2}, \tilde{x}_{i3}, \tilde{y}_i)$, $i = 1, 2, \cdots, 24$. For each i, $\tilde{x}_{i1}, \tilde{x}_{i2}, \tilde{x}_{i3}, \tilde{y}_i$ are independent linear uncertain variables. See Table 16.1. Let us show how the uncertain regression analysis is used to determine the functional relationship between (x_1, x_2, x_3) and y.

In order to determine it, we employ the uncertain linear regression model,

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon. \tag{16.70}$$

Table 16.1: 24 Imprecisely Observed Data

No.	x_1	x_2	x_3	y
1	$\mathcal{L}(3,4)$	$\mathcal{L}(9,10)$	$\mathcal{L}(6,7)$	$\mathcal{L}(33,36)$
2	$\mathcal{L}(5,6)$	$\mathcal{L}(20,22)$	$\mathcal{L}(6,7)$	$\mathcal{L}(40,43)$
3	$\mathcal{L}(5,6)$	$\mathcal{L}(18,20)$	$\mathcal{L}(7,8)$	$\mathcal{L}(38,41)$
4	$\mathcal{L}(5,6)$	$\mathcal{L}(33,36)$	$\mathcal{L}(6,7)$	$\mathcal{L}(46,49)$
5	$\mathcal{L}(4,5)$	$\mathcal{L}(31,34)$	$\mathcal{L}(7,8)$	$\mathcal{L}(41,44)$
6	$\mathcal{L}(6,7)$	$\mathcal{L}(13,15)$	$\mathcal{L}(5,6)$	$\mathcal{L}(37,40)$
7	$\mathcal{L}(6,7)$	$\mathcal{L}(25,28)$	$\mathcal{L}(6,7)$	$\mathcal{L}(39,42)$
8	$\mathcal{L}(5,6)$	$\mathcal{L}(30,33)$	$\mathcal{L}(4,5)$	$\mathcal{L}(40,43)$
9	$\mathcal{L}(3,4)$	$\mathcal{L}(5,6)$	$\mathcal{L}(5,6)$	$\mathcal{L}(30,33)$
10	$\mathcal{L}(7,8)$	$\mathcal{L}(47,50)$	$\mathcal{L}(8,9)$	$\mathcal{L}(52,55)$
11	$\mathcal{L}(4,5)$	$\mathcal{L}(25,28)$	$\mathcal{L}(5,6)$	$\mathcal{L}(38,41)$
12	$\mathcal{L}(4,5)$	$\mathcal{L}(11,13)$	$\mathcal{L}(6,7)$	$\mathcal{L}(31,34)$
13	$\mathcal{L}(8,9)$	$\mathcal{L}(23,26)$	$\mathcal{L}(7,8)$	$\mathcal{L}(43,46)$
14	$\mathcal{L}(6,7)$	$\mathcal{L}(35,38)$	$\mathcal{L}(7,8)$	$\mathcal{L}(44,47)$
15	$\mathcal{L}(6,7)$	$\mathcal{L}(39,44)$	$\mathcal{L}(5,6)$	$\mathcal{L}(42,45)$
16	$\mathcal{L}(3,4)$	$\mathcal{L}(21,24)$	$\mathcal{L}(4,5)$	$\mathcal{L}(33,36)$
17	$\mathcal{L}(6,7)$	$\mathcal{L}(7,8)$	$\mathcal{L}(5,6)$	$\mathcal{L}(34,37)$
18	$\mathcal{L}(7,8)$	$\mathcal{L}(40,43)$	$\mathcal{L}(7,8)$	$\mathcal{L}(48,51)$
19	$\mathcal{L}(4,5)$	$\mathcal{L}(35,38)$	$\mathcal{L}(6,7)$	$\mathcal{L}(38,41)$
20	$\mathcal{L}(4,5)$	$\mathcal{L}(23,26)$	$\mathcal{L}(3,4)$	$\mathcal{L}(35,38)$
21	$\mathcal{L}(5,6)$	$\mathcal{L}(33,36)$	$\mathcal{L}(4,5)$	$\mathcal{L}(40,43)$
22	$\mathcal{L}(5,6)$	$\mathcal{L}(27,30)$	$\mathcal{L}(4,5)$	$\mathcal{L}(36,39)$
23	$\mathcal{L}(4,5)$	$\mathcal{L}(34,37)$	$\mathcal{L}(8,9)$	$\mathcal{L}(45,48)$
24	$\mathcal{L}(3,4)$	$\mathcal{L}(15,17)$	$\mathcal{L}(5,6)$	$\mathcal{L}(35,38)$

By solving the minimization problem (16.44), we get the least squares estimate

$$(\beta_0^*, \beta_1^*, \beta_2^*, \beta_3^*) = (21.5196, 0.8678, 0.3110, 1.0053). \tag{16.71}$$

Thus the fitted linear regression model is

$$y = 21.5196 + 0.8678x_1 + 0.3110x_2 + 1.0053x_3. \tag{16.72}$$

By using the formulas (16.54) and (16.55), we get the expected value and variance of the disturbance term ε are

$$\hat{e} = 0.0000, \quad \hat{\sigma}^2 = 5.6305, \tag{16.73}$$

respectively. Now let

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \sim (\mathcal{L}(5,6), \mathcal{L}(28,30), \mathcal{L}(6,7)) \tag{16.74}$$

be a new uncertain explanatory vector. When $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \varepsilon$ are independent, by calculating the formula (16.62), we get the forecast value of response variable y is

$$\mu = 41.8460. \tag{16.75}$$

Taking the confidence level $\alpha = 95\%$, if the disturbance term ε is assumed to follow normal uncertainty distribution, then

$$b = 5.9780 \tag{16.76}$$

is the minimum value such that (16.66) holds. Therefore, the 95% confidence interval of response variable y is

$$41.8460 \pm 5.9780. \tag{16.77}$$

16.6 Uncertain Time Series Analysis

An uncertain time series is a sequence of imprecisely observed values that are characterized in terms of uncertain variables. Mathematically, an uncertain time series is represented by

$$X = \{X_1, X_2, \cdots, X_n\} \tag{16.78}$$

where X_t are imprecisely observed values (i.e., uncertain variables) at times t, $t = 1, 2, \cdots, n$, respectively. A basic problem of uncertain time series analysis is to predict the value of X_{n+1} based on previously observed values $X_1, X_2, \cdots, X_n$.

The simplest approach for modelling uncertain time series is the autoregressive model,

$$X_t = a_0 + \sum_{i=1}^{k} a_i X_{t-i} + \varepsilon_t \tag{16.79}$$

where $a_0, a_1, \cdots, a_k$ are unknown parameters, ε_t is a disturbance term, and k is called the order of the autoregressive model.

Based on the imprecisely observed values $X_1, X_2, \cdots, X_n$, Yang-Liu [163] suggested that the least squares estimate of $a_0, a_1, \cdots, a_k$ in the autoregressive model (16.79) is the solution of the minimization problem,

$$\min_{a_0, a_1, \cdots, a_k} \sum_{t=k+1}^{n} E\left[\left(X_t - a_0 - \sum_{i=1}^{k} a_i X_{t-i}\right)^2\right]. \tag{16.80}$$

If the minimization solution is $a_0^*, a_1^*, \cdots, a_k^*$, then the fitted autoregressive model is

$$X_t = a_0^* + \sum_{i=1}^{k} a_i^* X_{t-i}. \tag{16.81}$$

Theorem 16.3 *(Yang-Liu [163]) Let $X_1, X_2, \cdots, X_n$ be imprecisely observed values characterized in terms of independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Then the least squares estimate of $a_0, a_1, \cdots, a_k$ in the autoregressive model*

$$X_t = a_0 + \sum_{i=1}^{k} a_i X_{t-i} + \varepsilon_t \tag{16.82}$$

solves the minimization problem,

$$\min_{a_0, a_1, \cdots, a_k} \sum_{t=k+1}^{n} \int_0^1 \left(\Phi_t^{-1}(\alpha) - a_0 - \sum_{i=1}^{k} a_i \Upsilon_{t-i}^{-1}(\alpha) \right)^2 \mathrm{d}\alpha \tag{16.83}$$

where

$$\Upsilon_{t-i}^{-1}(\alpha) = \begin{cases} \Phi_{t-i}^{-1}(1-\alpha), & \text{if } a_i \ge 0 \\ \Phi_{t-i}^{-1}(\alpha), & \text{if } a_i < 0 \end{cases} \tag{16.84}$$

for $i = 1, 2, \cdots, k$.

Proof: For each index t, the inverse uncertainty distribution of the uncertain variable

$$X_t - a_0 - \sum_{i=1}^{k} a_i X_{t-i}$$

is just

$$F_t^{-1}(\alpha) = \Phi_t^{-1}(\alpha) - a_0 - \sum_{i=1}^{k} a_i \Upsilon_{t-i}^{-1}(\alpha).$$

It follows from Theorem 2.42 that

$$E\left[\left(X_t - a_0 - \sum_{i=1}^{k} a_i X_{t-i}\right)^2\right] = \int_0^1 \left(\Phi_t^{-1}(\alpha) - a_0 - \sum_{i=1}^{k} a_i \Upsilon_{t-i}^{-1}(\alpha)\right)^2 \mathrm{d}\alpha.$$

Hence the minimization problem (16.83) is equivalent to (16.80). The theorem is thus proved.

Residual Error

Definition 16.2 *(Yang-Liu [163]) Let $X_1, X_2, \cdots, X_n$ be imprecisely observed values, and let the fitted autoregressive model be*

$$X_t = a_0^* + \sum_{i=1}^{k} a_i^* X_{t-i}. \tag{16.85}$$

Then for each index t ($t = k+1, k+2, \cdots, n$), the difference between the actual observed value and the value predicted by the model,

$$\hat{\varepsilon}_t = X_t - a_0^* - \sum_{i=1}^{k} a_i^* X_{t-i} \tag{16.86}$$

is called the t-th residual error.

If disturbance terms $\varepsilon_{k+1}, \varepsilon_{k+2}, \cdots, \varepsilon_n$ are assumed to be iid uncertain variables (hereafter called iid hypothesis), then the expected value of disturbance terms can be estimated as the average of the expected values of residual errors, i.e.,

$$\hat{e} = \frac{1}{n-k} \sum_{t=k+1}^{n} E[\hat{\varepsilon}_t] \tag{16.87}$$

and the variance can be estimated as

$$\hat{\sigma}^2 = \frac{1}{n-k} \sum_{t=k+1}^{n} E[(\hat{\varepsilon}_t - \hat{e})^2] \tag{16.88}$$

where $\hat{\varepsilon}_t$ are the t-th residual errors, $t = k+1, k+2, \cdots, n$, respectively.

Theorem 16.4 *(Yang-Liu [163]) Let $X_1, X_2, \cdots, X_n$ be imprecisely observed values characterized in terms of independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and let the fitted autoregressive model be*

$$X_t = a_0^* + \sum_{i=1}^{k} a_i^* X_{t-i}. \tag{16.89}$$

Then the estimated expected value of disturbance terms under the iid hypothesis is

$$\hat{e} = \frac{1}{n-k} \sum_{t=k+1}^{n} \int_0^1 \left(\Phi_t^{-1}(\alpha) - a_0^* - \sum_{i=1}^{k} a_i^* \Upsilon_{t-i}^{-1}(\alpha) \right) \mathrm{d}\alpha \tag{16.90}$$

and the estimated variance is

$$\hat{\sigma}^2 = \frac{1}{n-k} \sum_{t=k+1}^{n} \int_0^1 \left(\Phi_t^{-1}(\alpha) - a_0^* - \sum_{i=1}^{k} a_i^* \Upsilon_{t-i}^{-1}(\alpha) - \hat{e} \right)^2 \mathrm{d}\alpha \tag{16.91}$$

where

$$\Upsilon_{t-i}^{-1}(\alpha) = \begin{cases} \Phi_{t-i}^{-1}(1-\alpha), & \textit{if } a_i^* \ge 0 \\ \Phi_{t-i}^{-1}(\alpha), & \textit{if } a_i^* < 0 \end{cases} \tag{16.92}$$

for $i = 1, 2, \cdots, k$.

Proof: For each index t, the inverse uncertainty distribution of the uncertain variable

$$X_t - a_0^* - \sum_{i=1}^{k} a_i^* X_{t-i}$$

is just

$$F_t^{-1}(\alpha) = \Phi_t^{-1}(\alpha) - a_0^* - \sum_{i=1}^{k} a_i^* \Upsilon_{t-i}^{-1}(\alpha).$$

It follows from Theorems 2.25 and 2.42 that (16.90) and (16.91) hold.

Forecast Value and Confidence Interval

Now let $X_1, X_2, \cdots, X_n$ be imprecisely observed values characterized in terms of independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Assume (i) the fitted autoregressive model is

$$X_t = a_0^* + \sum_{i=1}^{k} a_i^* X_{t-i}, \tag{16.93}$$

and (ii) the disturbance term ε_{n+1} has expected value $\hat{e}$ and variance $\hat{\sigma}^2$, and is independent of $X_1, X_2, \cdots, X_n$. Yang-Liu [163] suggested that the *forecast uncertain variable* of X_{n+1} based on $X_1, X_2, \cdots, X_n$ is determined by

$$\hat{X}_{n+1} = a_0^* + \sum_{i=1}^{k} a_i^* X_{n+1-i} + \varepsilon_{n+1}, \tag{16.94}$$

and the *forecast value* is defined as the expected value of the forecast uncertain variable $\hat{X}_{n+1}$, i.e.,

$$\mu = a_0^* + \sum_{i=1}^{k} a_i^* E[X_{n+1-i}] + \hat{e}. \tag{16.95}$$

If we suppose further that the disturbance term ε_{n+1} follows normal uncertainty distribution, then the inverse uncertainty distribution of forecast uncertain variable $\hat{X}_{n+1}$ is

$$\hat{\Phi}_{n+1}^{-1}(\alpha) = a_0^* + \sum_{i=1}^{k} a_i^* \Upsilon_{n+1-i}^{-1}(\alpha) + \Phi^{-1}(\alpha) \tag{16.96}$$

where

$$\Upsilon_{n+1-i}^{-1}(\alpha) = \begin{cases} \Phi_{n+1-i}^{-1}(\alpha), & \text{if } a_i^* \ge 0 \\ \Phi_{n+1-i}^{-1}(1-\alpha), & \text{if } a_i^* < 0 \end{cases} \tag{16.97}$$

for $i = 1, 2, \cdots, k$, and $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of $\mathcal{N}(\hat{e}, \hat{\sigma})$, i.e.,

$$\Phi^{-1}(\alpha) = \hat{e} + \frac{\hat{\sigma}\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{16.98}$$

From $\hat{\Phi}_{n+1}^{-1}$, we may also derive the uncertainty distribution $\hat{\Phi}_{n+1}$ of $\hat{X}_{n+1}$. Take α (e.g., 95%) as the confidence level, and find the minimum value b such that

$$\hat{\Phi}_{n+1}(\mu + b) - \hat{\Phi}_{n+1}(\mu - b) \geq \alpha. \tag{16.99}$$

Since $\mathcal{M}\{\mu - b \leq \hat{X}_{n+1} \leq \mu + b\} \geq \hat{\Phi}_{n+1}(\mu + b) - \hat{\Phi}_{n+1}(\mu - b) \geq \alpha$, Yang-Liu [163] suggested that the α *confidence interval* of X_{n+1} is

$$\mu \pm b. \tag{16.100}$$

Exercise 16.5: Let $X_1, X_2, \cdots, X_n$ be imprecisely observed values characterized in terms of independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Assume (i) the fitted autoregressive model is

$$X_t = a_0^* + \sum_{i=1}^{k} a_i^* X_{t-i}, \tag{16.101}$$

and (ii) the disturbance term ε_{n+1} follows linear uncertainty distribution with expected value $\hat{e}$ and variance $\hat{\sigma}^2$, and is independent of $X_1, X_2, \cdots, X_n$. What is the α confidence interval of X_{n+1}? (Hint: The linear uncertain variable $\mathcal{L}(\hat{e} - \sqrt{3}\hat{\sigma}, \hat{e} + \sqrt{3}\hat{\sigma})$ has expected value $\hat{e}$ and variance $\hat{\sigma}^2$.)

Example 16.7: Assume there exist 20 imprecisely observed carbon emissions $X_1, X_2, \cdots, X_{20}$ that are characterized in terms of independent linear uncertain variables. See Table 16.2. Let us show how the uncertain time series analysis is used to forecast the carbon emission in the 21st year.

Table 16.2: Imprecisely Observed Carbon Emissions over 20 Years

X_1	X_2	X_3	X_4	X_5
$\mathcal{L}(330, 341)$	$\mathcal{L}(333, 346)$	$\mathcal{L}(335, 347)$	$\mathcal{L}(338, 350)$	$\mathcal{L}(340, 354)$
X_6	X_7	X_8	X_9	X_{10}
$\mathcal{L}(343, 359)$	$\mathcal{L}(344, 364)$	$\mathcal{L}(346, 366)$	$\mathcal{L}(350, 366)$	$\mathcal{L}(355, 369)$
X_{11}	X_{12}	X_{13}	X_{14}	X_{15}
$\mathcal{L}(360, 372)$	$\mathcal{L}(362, 376)$	$\mathcal{L}(365, 381)$	$\mathcal{L}(370, 384)$	$\mathcal{L}(373, 390)$
X_{16}	X_{17}	X_{18}	X_{19}	X_{20}
$\mathcal{L}(379, 391)$	$\mathcal{L}(380, 398)$	$\mathcal{L}(384, 402)$	$\mathcal{L}(388, 410)$	$\mathcal{L}(390, 415)$

In order to forecast it, we employ the 2-order uncertain autoregressive model,

$$X_t = a_0 + a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t. \tag{16.102}$$

By solving the minimization problem (16.83), we get the least squares estimate

$$(a_0^*, a_1^*, a_2^*) = (28.4715, 0.2367, 0.7018). \tag{16.103}$$

Thus the fitted autoregressive model is

$$X_t = 28.4715 + 0.2367 X_{t-1} + 0.7018 X_{t-2}. \tag{16.104}$$

By using the formulas (16.90) and (16.91), we get the expected value and variance of disturbance term ε_{21} are

$$\hat{e} = 0.0000, \quad \hat{\sigma}^2 = 84.7422, \tag{16.105}$$

respectively. When the disturbance term ε_{21} is assumed to be independent of X_{20} and X_{19}, by calculating the formula (16.95), we get the forecast value of carbon emission in the 21st year (i.e., X_{21}) is

$$\mu = 403.7361. \tag{16.106}$$

Taking the confidence level $\alpha = 95\%$, if the disturbance term ε_{21} is assumed to follow normal uncertainty distribution, then

$$b = 28.7376 \tag{16.107}$$

is the minimum value such that (16.99) holds. Therefore, the 95% confidence interval of carbon emission in the 21st year (i.e., X_{21}) is

$$403.7361 \pm 28.7376. \tag{16.108}$$

16.7 Bibliographic Notes

The study of uncertain statistics was started by Liu [83] in 2010 in which a questionnaire survey for collecting expert's experimental data was designed. It was showed among others by Chen-Ralescu [11] that the questionnaire survey may successfully acquire the expert's experimental data.

Parametric uncertain statistics assumes that the uncertainty distribution to be determined has a known functional form but with unknown parameters. In order to estimate the unknown parameters, Liu [83] suggested the principle of least squares, and Wang-Peng [152] proposed the method of moments. Nonparametric uncertain statistics does not rely on the expert's experimental data belonging to any particular uncertainty distribution. In order to determine the uncertainty distributions, Liu [83] introduced the linear interpolation method (i.e., empirical uncertainty distribution), and Chen-Ralescu

[11] proposed a series of spline interpolation methods. When multiple domain experts are available, Wang-Gao-Guo [150] recast Delphi method as a process to determine uncertainty distributions.

In order to determine membership functions, a questionnaire survey for collecting expert's experimental data was designed by Liu [84]. Based on expert's experimental data, Liu [84] also suggested the linear interpolation method and the principle of least squares to determine membership functions. When multiple domain experts are available, Delphi method was introduced to uncertain statistics by Guo-Wang-Wang-Chen [52].

Uncertain regression analysis is used to model the relationship between explanatory variables and response variables when the imprecise observations are characterized in terms of uncertain variables. For that matter, Yao-Liu [184] suggested the principle of least squares to estimate the unknown parameters in the regression models. Lio-Liu [73] analyzed the residual error and confidence interval of forecast values.

Uncertain time series analysis was first presented by Yang-Liu [163] in order to predict the future values based on preciously imprecise observations that are characterized in terms of uncertain variables.

Appendix A

Uncertain Random Variable

Uncertainty and randomness are two basic types of indeterminacy. Uncertain random variable was initialized by Liu [105] in 2013 for modelling complex systems with not only uncertainty but also randomness. This appendix will introduce the concepts of chance measure, uncertain random variable, chance distribution, operational law, expected value, variance, and law of large numbers. As applications of chance theory, this appendix will also provide uncertain random programming, uncertain random risk analysis, uncertain random reliability analysis, uncertain random graph, uncertain random network, and uncertain random process.

A.1 Chance Measure

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let $(\Omega, \mathcal{A}, \Pr)$ be a probability space. Then the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is called a *chance space*. Essentially, it is another triplet,

$$(\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \Pr) \tag{A.1}$$

where $\Gamma \times \Omega$ is the universal set, $\mathcal{L} \times \mathcal{A}$ is the product σ-algebra, and $\mathcal{M} \times \Pr$ is the product measure.

The universal set $\Gamma \times \Omega$ is clearly the set of all ordered pairs of the form (γ, ω), where $\gamma \in \Gamma$ and $\omega \in \Omega$. That is,

$$\Gamma \times \Omega = \{(\gamma, \omega) \mid \gamma \in \Gamma, \omega \in \Omega\}. \tag{A.2}$$

The product σ-algebra $\mathcal{L} \times \mathcal{A}$ is the smallest σ-algebra containing measurable rectangles of the form $\Lambda \times A$, where $\Lambda \in \mathcal{L}$ and $A \in \mathcal{A}$. Any element in $\mathcal{L} \times \mathcal{A}$ is called an *event* in the chance space.

What is the product measure $\mathcal{M} \times \Pr$? In order to answer this question, let us consider an event Θ in $\mathcal{L} \times \mathcal{A}$. For each $\omega \in \Omega$, the cross section

$$\Theta_\omega = \{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \tag{A.3}$$

is clearly an event in $\mathcal{L}$. Thus the uncertain measure of Θ_ω, i.e.,

$$\mathcal{M}\{\Theta_\omega\} = \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \tag{A.4}$$

exists for each $\omega \in \Omega$. If $\mathcal{M}\{\Theta_\omega\}$ is measurable with respect to ω, then it is a random variable. Now we define $\mathcal{M} \times \Pr$ of Θ as the average value of $\mathcal{M}\{\Theta_\omega\}$ in the sense of probability measure (i.e., the expected value), and call it chance measure represented by $\mathrm{Ch}\{\Theta\}$.

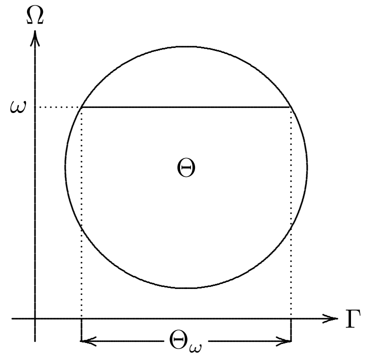


Figure A.1: An Event Θ in $\mathcal{L} \times \mathcal{A}$ and its Cross Section Θ_ω

Definition A.1 *(Liu [105]) Let* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ *be a chance space, and let* $\Theta \in \mathcal{L} \times \mathcal{A}$ *be an event. Then the chance measure of* Θ *is defined as*

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq x\} \,\mathrm{d}x. \tag{A.5}$$

Exercise A.1: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure, and take a probability space $(\Omega, \mathcal{A}, \Pr)$ to be also $[0, 1]$ with Borel algebra and Lebesgue measure. Then

$$\Theta = \{(\gamma, \omega) \in \Gamma \times \Omega \mid \gamma + \omega \leq 1\} \tag{A.6}$$

is an event on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$. Show that

$$\mathrm{Ch}\{\Theta\} = \frac{1}{2}. \tag{A.7}$$

Exercise A.2: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure, and take a probability space $(\Omega, \mathcal{A}, \Pr)$ to be also $[0,1]$ with Borel algebra and Lebesgue measure. Then

$$\Theta = \left\{(\gamma, \omega) \in \Gamma \times \Omega \,|\, (\gamma - 0.5)^2 + (\omega - 0.5)^2 \le 0.5^2\right\} \tag{A.8}$$

is an event on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$. Show that

$$\mathrm{Ch}\{\Theta\} = \frac{\pi}{4}. \tag{A.9}$$

Theorem A.1 *(Liu [105]) Let* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ *be a chance space. Then*

$$\mathrm{Ch}\{\Lambda \times A\} = \mathcal{M}\{\Lambda\} \times \Pr\{A\} \tag{A.10}$$

for any $\Lambda \in \mathcal{L}$ *and any* $A \in \mathcal{A}$. *Especially, we have*

$$\mathrm{Ch}\{\emptyset\} = 0, \quad \mathrm{Ch}\{\Gamma \times \Omega\} = 1. \tag{A.11}$$

Proof: Let us first prove the identity (A.10). When A is nonempty, we have

$$\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Lambda \times A\} = \Lambda$$

and

$$\mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Lambda \times A\} = \mathcal{M}\{\Lambda\}.$$

For any real number x, if $\mathcal{M}\{\Lambda\} \ge x$, then

$$\Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Lambda \times A\} \ge x\} = \Pr\{A\}.$$

If $\mathcal{M}\{\Lambda\} < x$, then

$$\Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Lambda \times A\} \ge x\} = \Pr\{\emptyset\} = 0.$$

Thus

$$\begin{aligned}
\mathrm{Ch}\{\Lambda \times A\} &= \int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Lambda \times A\} \ge x\}\, \mathrm{d}x \\
&= \int_0^{\mathcal{M}\{\Lambda\}} \Pr\{A\} \mathrm{d}x + \int_{\mathcal{M}\{\Lambda\}}^1 0 \mathrm{d}x = \mathcal{M}\{\Lambda\} \times \Pr\{A\}.
\end{aligned}$$

Furthermore, it follows from (A.10) that

$$\mathrm{Ch}\{\emptyset\} = \mathcal{M}\{\emptyset\} \times \Pr\{\emptyset\} = 0,$$

$$\mathrm{Ch}\{\Gamma \times \Omega\} = \mathcal{M}\{\Gamma\} \times \Pr\{\Omega\} = 1.$$

The theorem is thus verified.

Theorem A.2 *(Liu [105], Monotonicity Theorem) The chance measure is a monotone increasing set function. That is, for any events* Θ_1 *and* Θ_2 *with* $\Theta_1 \subset \Theta_2$*, we have*

$$\mathrm{Ch}\{\Theta_1\} \le \mathrm{Ch}\{\Theta_2\}. \tag{A.12}$$

Proof: Since Θ_1 and Θ_2 are two events with $\Theta_1 \subset \Theta_2$, we immediately have

$$\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta_1\} \subset \{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta_2\}$$

and

$$\mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta_1\} \le \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta_2\}.$$

Thus for any real number x, we have

$$\begin{aligned} &\Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta_1\} \ge x\} \\ &\quad \le \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta_2\} \ge x\}. \end{aligned}$$

By the definition of chance measure, we get

$$\begin{aligned} \mathrm{Ch}\{\Theta_1\} &= \int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta_1\} \ge x\}\,\mathrm{d}x \\ &\le \int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta_2\} \ge x\}\,\mathrm{d}x = \mathrm{Ch}\{\Theta_2\}. \end{aligned}$$

That is, $\mathrm{Ch}\{\Theta\}$ is a monotone increasing function with respect to Θ. The theorem is thus verified.

Theorem A.3 *(Liu [105], Duality Theorem) The chance measure is self-dual. That is, for any event* Θ*, we have*

$$\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1. \tag{A.13}$$

Proof: Since both uncertain measure and probability measure are self-dual, we have

$$\begin{aligned} \mathrm{Ch}\{\Theta\} &= \int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta\} \ge x\}\,\mathrm{d}x \\ &= \int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta^c\} \le 1 - x\}\,\mathrm{d}x \\ &= \int_0^1 (1 - \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta^c\} > 1 - x\})\,\mathrm{d}x \\ &= 1 - \int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, (\gamma, \omega) \in \Theta^c\} > x\}\,\mathrm{d}x \\ &= 1 - \mathrm{Ch}\{\Theta^c\}. \end{aligned}$$

That is, $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1$, i.e., the chance measure is self-dual.

Theorem A.4 *(Hou [54], Subadditivity Theorem) The chance measure is subadditive. That is, for any countable sequence of events $\Theta_1, \Theta_2, \cdots$, we have*

$$\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty}\Theta_i\right\} \le \sum_{i=1}^{\infty}\mathrm{Ch}\{\Theta_i\}. \tag{A.14}$$

Proof: At first, it follows from the subadditivity of uncertain measure that

$$\mathcal{M}\left\{\gamma\in\Gamma \,|\, (\gamma,\omega)\in\bigcup_{i=1}^{\infty}\Theta_i\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\gamma\in\Gamma \,|\, (\gamma,\omega)\in\Theta_i\}.$$

Thus for any real number x, we have

$$\begin{aligned}
&\Pr\left\{\omega\in\Omega \,|\, \mathcal{M}\left\{\gamma\in\Gamma \,|\, (\gamma,\omega)\in\bigcup_{i=1}^{\infty}\Theta_i\right\}\ge x\right\}\\
&\le \Pr\left\{\omega\in\Omega \,|\, \sum_{i=1}^{\infty}\mathcal{M}\{\gamma\in\Gamma \,|\, (\gamma,\omega)\in\Theta_i\}\ge x\right\}.
\end{aligned}$$

By the definition of chance measure, we get

$$\begin{aligned}
\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty}\Theta_i\right\} &= \int_0^1 \Pr\left\{\omega\in\Omega \,|\, \mathcal{M}\left\{\gamma\in\Gamma \,|\, (\gamma,\omega)\in\bigcup_{i=1}^{\infty}\Theta_i\right\}\ge x\right\}\mathrm{d}x\\
&\le \int_0^1 \Pr\left\{\omega\in\Omega \,|\, \sum_{i=1}^{\infty}\mathcal{M}\{\gamma\in\Gamma \,|\, (\gamma,\omega)\in\Theta_i\}\ge x\right\}\mathrm{d}x\\
&\le \int_0^{+\infty} \Pr\left\{\omega\in\Omega \,|\, \sum_{i=1}^{\infty}\mathcal{M}\{\gamma\in\Gamma \,|\, (\gamma,\omega)\in\Theta_i\}\ge x\right\}\mathrm{d}x\\
&= \sum_{i=1}^{\infty}\int_0^1 \Pr\left\{\omega\in\Omega \,|\, \mathcal{M}\{\gamma\in\Gamma \,|\, (\gamma,\omega)\in\Theta_i\}\ge x\right\}\mathrm{d}x\\
&= \sum_{i=1}^{\infty}\mathrm{Ch}\{\Theta_i\}.
\end{aligned}$$

That is, the chance measure is subadditive.

A.2 Uncertain Random Variable

Theoretically, an uncertain random variable is a measurable function on the chance space. It is usually used to deal with measurable functions of uncertain variables and random variables.

Definition A.2 *(Liu [105]) An uncertain random variable is a function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B of real numbers.*

Remark A.1: An uncertain random variable $\xi(\gamma,\omega)$ degenerates to a random variable if it does not vary with γ. Thus a random variable is a special uncertain random variable.

Remark A.2: An uncertain random variable $\xi(\gamma,\omega)$ degenerates to an uncertain variable if it does not vary with ω. Thus an uncertain variable is a special uncertain random variable.

Theorem A.5 *Let $\xi_1,\xi_2,\cdots,\xi_n$ be uncertain random variables on the chance space $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$, and let f be a measurable function. Then*

$$\xi = f(\xi_1,\xi_2,\cdots,\xi_n) \tag{A.15}$$

is an uncertain random variable determined by

$$\xi(\gamma,\omega) = f(\xi_1(\gamma,\omega),\xi_2(\gamma,\omega),\cdots,\xi_n(\gamma,\omega)) \tag{A.16}$$

for all $(\gamma,\omega)\in\Gamma\times\Omega$.

Proof: Since $\xi_1,\xi_2,\cdots,\xi_n$ are uncertain random variables, we know that they are measurable functions on the chance space, and $\xi = f(\xi_1,\xi_2,\cdots,\xi_n)$ is also a measurable function. Hence ξ is an uncertain random variable.

Example A.1: A random variable η plus an uncertain variable τ makes an uncertain random variable ξ, i.e.,

$$\xi(\gamma,\omega) = \eta(\omega) + \tau(\gamma) \tag{A.17}$$

for all $(\gamma,\omega)\in\Gamma\times\Omega$.

Example A.2: A random variable η times an uncertain variable τ makes an uncertain random variable ξ, i.e.,

$$\xi(\gamma,\omega) = \eta(\omega)\cdot\tau(\gamma) \tag{A.18}$$

for all $(\gamma,\omega)\in\Gamma\times\Omega$.

Theorem A.6 *(Liu [105]) Let ξ be an uncertain random variable on the chance space $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$, and let B be a Borel set of real numbers. Then $\{\xi\in B\}$ is an uncertain random event with chance measure*

$$\mathrm{Ch}\{\xi\in B\} = \int_0^1 \Pr\left\{\omega\in\Omega \,|\, \mathcal{M}\{\gamma\in\Gamma \,|\, \xi(\gamma,\omega)\in B\}\ge x\right\}\mathrm{d}x. \tag{A.19}$$

Proof: Since $\{\xi\in B\}$ is an event in the chance space, the equation (A.19) follows from Definition A.1 immediately.

Remark A.3: If the uncertain random variable degenerates to a random variable η, then $\mathrm{Ch}\{\eta\in B\} = \mathrm{Ch}\{\Gamma\times(\eta\in B)\} = \mathcal{M}\{\Gamma\}\times\Pr\{\eta\in B\} = \Pr\{\eta\in B\}$. That is,

$$\mathrm{Ch}\{\eta\in B\} = \Pr\{\eta\in B\}. \tag{A.20}$$

If the uncertain random variable degenerates to an uncertain variable τ, then $\text{Ch}\{\tau \in B\} = \text{Ch}\{(\tau \in B) \times \Omega\} = \mathcal{M}\{\tau \in B\} \times \Pr\{\Omega\} = \mathcal{M}\{\tau \in B\}$. That is,

$$\text{Ch}\{\tau \in B\} = \mathcal{M}\{\tau \in B\}. \tag{A.21}$$

Theorem A.7 *(Liu [105]) Let ξ be an uncertain random variable. Then the chance measure* $\text{Ch}\{\xi \in B\}$ *is a monotone increasing function of B and*

$$\text{Ch}\{\xi \in \emptyset\} = 0, \quad \text{Ch}\{\xi \in \Re\} = 1. \tag{A.22}$$

Proof: Let B_1 and B_2 be Borel sets of real numbers with $B_1 \subset B_2$. Then we immediately have $\{\xi \in B_1\} \subset \{\xi \in B_2\}$. It follows from the monotonicity of chance measure that

$$\text{Ch}\{\xi \in B_1\} \le \text{Ch}\{\xi \in B_2\}.$$

Hence $\text{Ch}\{\xi \in B\}$ is a monotone increasing function of B. Furthermore, we have

$$\text{Ch}\{\xi \in \emptyset\} = \text{Ch}\{\emptyset\} = 0,$$
$$\text{Ch}\{\xi \in \Re\} = \text{Ch}\{\Gamma \times \Omega\} = 1.$$

The theorem is verified.

Theorem A.8 *(Liu [105]) Let ξ be an uncertain random variable. Then for any Borel set B of real numbers, we have*

$$\text{Ch}\{\xi \in B\} + \text{Ch}\{\xi \in B^c\} = 1. \tag{A.23}$$

Proof: It follows from $\{\xi \in B\}^c = \{\xi \in B^c\}$ and the duality of chance measure immediately.

A.3 Chance Distribution

Definition A.3 *(Liu [105]) Let ξ be an uncertain random variable. Then its chance distribution is defined by*

$$\Phi(x) = \text{Ch}\{\xi \le x\} \tag{A.24}$$

for any $x \in \Re$.

Example A.3: As a special uncertain random variable, the chance distribution of a random variable η is just its probability distribution, that is,

$$\Phi(x) = \text{Ch}\{\eta \le x\} = \Pr\{\eta \le x\}. \tag{A.25}$$

Example A.4: As a special uncertain random variable, the chance distribution of an uncertain variable τ is just its uncertainty distribution, that is,

$$\Phi(x) = \text{Ch}\{\tau \le x\} = \mathcal{M}\{\tau \le x\}. \tag{A.26}$$

Theorem A.9 *(Liu [105], Sufficient and Necessary Condition for Chance Distribution) A function $\Phi : \Re \to [0,1]$ is a chance distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.*

Proof: Assume Φ is a chance distribution of uncertain random variable ξ. Let x_1 and x_2 be two real numbers with $x_1 < x_2$. It follows from Theorem A.7 that

$$\Phi(x_1) = \mathrm{Ch}\{\xi \le x_1\} \le \mathrm{Ch}\{\xi \le x_2\} = \Phi(x_2).$$

Hence the chance distribution Φ is a monotone increasing function. Furthermore, if $\Phi(x) \equiv 0$, then

$$\int_0^1 \Pr\left\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma,\omega) \le x\} \ge r\right\} \mathrm{d}r \equiv 0.$$

Thus for almost all $\omega \in \Omega$, we have

$$\mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma,\omega) \le x\} \equiv 0, \quad \forall x \in \Re$$

which is in contradiction to the asymptotic theorem, and then $\Phi(x) \not\equiv 0$ is verified. Similarly, if $\Phi(x) \equiv 1$, then

$$\int_0^1 \Pr\left\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma,\omega) \le x\} \ge r\right\} \mathrm{d}r \equiv 1.$$

Thus for almost all $\omega \in \Omega$, we have

$$\mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma,\omega) \le x\} \equiv 1, \quad \forall x \in \Re$$

which is also in contradiction to the asymptotic theorem, and then $\Phi(x) \not\equiv 1$ is proved.

Conversely, suppose $\Phi : \Re \to [0,1]$ is a monotone increasing function but $\Phi(x) \not\equiv 0$ and $\Phi(x) \not\equiv 1$. It follows from Peng-Iwamura theorem that there is an uncertain variable whose uncertainty distribution is just $\Phi(x)$. Since an uncertain variable is a special uncertain random variable, we know that Φ is a chance distribution.

Theorem A.10 *(Liu [105], Chance Inversion Theorem) Let ξ be an uncertain random variable with chance distribution Φ. Then for any real number x, we have*

$$\mathrm{Ch}\{\xi \le x\} = \Phi(x), \quad \mathrm{Ch}\{\xi > x\} = 1 - \Phi(x). \tag{A.27}$$

Proof: The equation $\mathrm{Ch}\{\xi \le x\} = \Phi(x)$ follows from the definition of chance distribution immediately. By using the duality of chance measure, we get

$$\mathrm{Ch}\{\xi > x\} = 1 - \mathrm{Ch}\{\xi \le x\} = 1 - \Phi(x).$$

Remark A.4: When the chance distribution Φ is a continuous function, we also have

$$\mathrm{Ch}\{\xi < x\} = \Phi(x), \quad \mathrm{Ch}\{\xi \ge x\} = 1 - \Phi(x). \tag{A.28}$$

A.4 Operational Law

Assume $\eta_1, \eta_2, \cdots, \eta_m$ are independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and $\tau_1, \tau_2, \cdots, \tau_n$ are independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. What is the chance distribution of the uncertain random variable

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)? \tag{A.29}$$

This section will provide an operational law to answer this question.

Theorem A.11 *(Liu [106]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables. Assume f is a measurable function. Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) \tag{A.30}$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, y_2, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \cdots \mathrm{d}\Psi_m(y_m) \tag{A.31}$$

where $F(x; y_1, y_2, \cdots, y_m)$ is the uncertainty distribution of the uncertain variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$.

Proof: It follows from Theorem A.6 that the uncertain random variable ξ has a chance distribution

$$\begin{aligned}
\Phi(x) &= \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid \xi(\gamma, \omega) \le x\} \ge r\} \, \mathrm{d}r \\
&= \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{f(\eta_1(\omega), \cdots, \eta_m(\omega), \tau_1, \cdots, \tau_n) \le x\} \ge r\} \, \mathrm{d}r \\
&= \int_{\Re^m} \mathcal{M}\{f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n) \le x\} \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \\
&= \int_{\Re^m} F(x; y_1, y_2, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \cdots \mathrm{d}\Psi_m(y_m)
\end{aligned}$$

where $F(x; y_1, y_2, \cdots, y_m)$ is just the uncertainty distribution of the uncertain variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$. The theorem is thus verified.

Exercise A.3: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Show that the sum

$$\xi = \eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n \tag{A.32}$$

has a chance distribution

$$\Phi(x) = \int_{-\infty}^{+\infty} \Upsilon(x-y)\mathrm{d}\Psi(y) \tag{A.33}$$

where

$$\Psi(y) = \int_{y_1+y_2+\cdots+y_m\le y} \mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\cdots\mathrm{d}\Psi_m(y_m) \tag{A.34}$$

is the probability distribution of $\eta_1+\eta_2+\cdots+\eta_m$, and

$$\Upsilon(z) = \sup_{z_1+z_2+\cdots+z_n=z} \Upsilon_1(z_1)\wedge\Upsilon_2(z_2)\wedge\cdots\wedge\Upsilon_n(z_n) \tag{A.35}$$

is the uncertainty distribution of $\tau_1+\tau_2+\cdots+\tau_n$.

Exercise A.4: Let $\eta_1,\eta_2,\cdots,\eta_m$ be independent positive random variables with probability distributions $\Psi_1,\Psi_2,\cdots,\Psi_m$, and let $\tau_1,\tau_2,\cdots,\tau_n$ be independent positive uncertain variables with uncertainty distributions $\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_n$, respectively. Show that the product

$$\xi = \eta_1\eta_2\cdots\eta_m\tau_1\tau_2\cdots\tau_n \tag{A.36}$$

has a chance distribution

$$\Phi(x) = \int_{0}^{+\infty} \Upsilon(x/y)\mathrm{d}\Psi(y) \tag{A.37}$$

where

$$\Psi(y) = \int_{y_1y_2\cdots y_m\le y} \mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\cdots\mathrm{d}\Psi_m(y_m) \tag{A.38}$$

is the probability distribution of $\eta_1\eta_2\cdots\eta_m$, and

$$\Upsilon(z) = \sup_{z_1z_2\cdots z_n=z} \Upsilon_1(z_1)\wedge\Upsilon_2(z_2)\wedge\cdots\wedge\Upsilon_n(z_n) \tag{A.39}$$

is the uncertainty distribution of $\tau_1\tau_2\cdots\tau_n$.

Exercise A.5: Let $\eta_1,\eta_2,\cdots,\eta_m$ be independent random variables with probability distributions $\Psi_1,\Psi_2,\cdots,\Psi_m$, and let $\tau_1,\tau_2,\cdots,\tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_n$, respectively. Show that the minimum

$$\xi = \eta_1\wedge\eta_2\wedge\cdots\wedge\eta_m\wedge\tau_1\wedge\tau_2\wedge\cdots\wedge\tau_n \tag{A.40}$$

has a chance distribution

$$\Phi(x) = \Psi(x)+\Upsilon(x)-\Psi(x)\Upsilon(x) \tag{A.41}$$

where

$$\Psi(x) = 1 - (1 - \Psi_1(x))(1 - \Psi_2(x)) \cdots (1 - \Psi_m(x)) \tag{A.42}$$

is the probability distribution of $\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m$, and

$$\Upsilon(x) = \Upsilon_1(x) \vee \Upsilon_2(x) \vee \cdots \vee \Upsilon_n(x) \tag{A.43}$$

is the uncertainty distribution of $\tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n$.

Exercise A.6: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Show that the maximum

$$\xi = \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n \tag{A.44}$$

has a chance distribution

$$\Phi(x) = \Psi(x)\Upsilon(x) \tag{A.45}$$

where

$$\Psi(x) = \Psi_1(x)\Psi_2(x) \cdots \Psi_m(x) \tag{A.46}$$

is the probability distribution of $\eta_1 \vee \eta_2 \vee \cdots \vee \eta_m$, and

$$\Upsilon(x) = \Upsilon_1(x) \wedge \Upsilon_2(x) \wedge \cdots \wedge \Upsilon_n(x) \tag{A.47}$$

is the uncertainty distribution of $\tau_1 \vee \tau_2 \vee \cdots \vee \tau_n$.

Theorem A.12 *(Liu [106]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with continuous uncertainty distributions Υ_1, $\Upsilon_2, \cdots, \Upsilon_n$, respectively. Assume $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \tau_{k+2}, \cdots, \tau_n$. Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) \tag{A.48}$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, y_2, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \cdots \mathrm{d}\Psi_m(y_m) \tag{A.49}$$

where

$$F(x; y_1, \cdots, y_m) = \sup_{f(y_1, \cdots, y_m, x_1, \cdots, x_n) = x} \left(\min_{1 \le i \le k} \Upsilon_i(x_i) \wedge \min_{k+1 \le i \le n} (1 - \Upsilon_i(x_i)) \right).$$

Proof: It follows from Theorems 2.20 and A.11 immediately.

Theorem A.13 *(Liu [106]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Assume $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \tau_{k+2}, \cdots, \tau_n$. Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) \tag{A.50}$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, y_2, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \cdots \mathrm{d}\Psi_m(y_m) \tag{A.51}$$

where $F(x; y_1, y_2, \cdots, y_m)$ is the root α of the equation

$$f(y_1, y_2, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) = x.$$

Proof: It follows from Theorem 2.14 that $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$G^{-1}(\alpha) = f(y_1, y_2, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)).$$

Since $\mathcal{M}\{f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n) \leq x\} = G(x)$, it is the solution α of the equation $G^{-1}(\alpha) = x$. Thus (A.51) follows from Theorem A.11 immediately.

Remark A.5: Sometimes, the equation in the above theorem may not have a root. In this case, if

$$f(y_1, y_2, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) < x$$

for all α, then we set the root $\alpha = 1$; and if

$$f(y_1, y_2, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) > x$$

for all α, then we set the root $\alpha = 0$. The root α may be estimated by the bisection method because

$$f(y_1, y_2, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha))$$

is a strictly increasing function with respect to α.

Order Statistics

Definition A.4 *(Gao-Sun-Ralescu [37], Order Statistic) Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain random variables, and let k be an index with $1 \leq k \leq n$. Then*

$$\xi = k\text{-}\min[\xi_1, \xi_2, \cdots, \xi_n] \tag{A.52}$$

is called the kth order statistic of $\xi_1, \xi_2, \cdots, \xi_n$.

Theorem A.14 *(Gao-Sun-Ralescu [37]) Let $\eta_1, \eta_2, \cdots, \eta_n$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_n$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $f_1, f_2, \cdots, f_n$ are continuous and strictly increasing functions, then the kth order statistic of $f_1(\eta_1, \tau_1), f_2(\eta_2, \tau_2), \cdots, f_n(\eta_n, \tau_n)$ has a chance distribution*

$$\Phi(x) = \int_{\Re^n} k\text{-max} \begin{bmatrix} \sup\limits_{f_1(y_1,z_1)=x} \Upsilon_1(z_1) \\ \sup\limits_{f_2(y_2,z_2)=x} \Upsilon_2(z_2) \\ \cdots \\ \sup\limits_{f_n(y_n,z_n)=x} \Upsilon_n(z_n) \end{bmatrix} \mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\cdots\mathrm{d}\Psi_n(y_n).$$

Proof: For each index i and each real number y_i, since f_i is a strictly increasing function, the uncertain variable $f_i(y_i, \tau_i)$ has an uncertainty distribution

$$F_i(x; y_i) = \sup_{f_i(y_i,z_i)=x} \Upsilon_i(z_i).$$

Theorem 2.17 states that the kth order statistic of $f_1(y_1, \tau_1), f_2(y_2, \tau_2), \cdots, f_n(y_n, \tau_n)$ has an uncertainty distribution

$$F(x; y_1, y_2, \cdots, y_n) = k\text{-max}\left[\sup_{f_1(y_1,z_1)=x} \Upsilon_1(z_1), \cdots, \sup_{f_n(y_n,z_n)=x} \Upsilon_n(z_n)\right].$$

Thus the theorem follows from the operational law of uncertain random variables immediately.

Exercise A.7: Let $\eta_1, \eta_2, \cdots, \eta_n$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_n$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with continuous uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Assume $f_1, f_2, \cdots, f_n$ are continuous and strictly decreasing functions. Show that the kth order statistic of $f_1(\eta_1, \tau_1), f_2(\eta_2, \tau_2), \cdots, f_n(\eta_n, \tau_n)$ has a chance distribution

$$\Phi(x) = \int_{\Re^n} k\text{-max} \begin{bmatrix} \sup\limits_{f_1(y_1,z_1)=x} (1 - \Upsilon_1(z_1)) \\ \sup\limits_{f_2(y_2,z_2)=x} (1 - \Upsilon_2(z_2)) \\ \cdots \\ \sup\limits_{f_n(y_n,z_n)=x} (1 - \Upsilon_n(z_n)) \end{bmatrix} \mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\cdots\mathrm{d}\Psi_n(y_n).$$

Exercise A.8: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Show that the kth order statistic of $\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} k\text{-max}\begin{bmatrix} I(y_1 \le x), \cdots, I(y_m \le x) \\ \Upsilon_1(x), \Upsilon_2(x), \cdots, \Upsilon_n(x) \end{bmatrix} \mathrm{d}\Psi_1(y_1)\cdots \mathrm{d}\Psi_m(y_m). \quad (A.53)$$

Operational Law for Boolean System

Theorem A.15 *(Liu [106]) Assume $\eta_1, \eta_2, \cdots, \eta_m$ are independent Boolean random variables, i.e.,*

$$\eta_i = \begin{cases} 1 \text{ with probability measure } a_i \\ 0 \text{ with probability measure } 1 - a_i \end{cases} \quad (A.54)$$

for $i = 1, 2, \cdots, m$, and $\tau_1, \tau_2, \cdots, \tau_n$ are independent Boolean uncertain variables, i.e.,

$$\tau_j = \begin{cases} 1 \text{ with uncertain measure } b_j \\ 0 \text{ with uncertain measure } 1 - b_j \end{cases} \quad (A.55)$$

for $j = 1, 2, \cdots, n$. If f is a Boolean function, then

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \quad (A.56)$$

is a Boolean uncertain random variable such that

$$\mathrm{Ch}\{\xi = 1\} = \sum_{(x_1, \cdots, x_m) \in \{0,1\}^m} \left(\prod_{i=1}^m \mu_i(x_i) \right) f^*(x_1, \cdots, x_m) \quad (A.57)$$

where

$$f^*(x_1, \cdots, x_m) = \begin{cases} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1 \le j \le n} \nu_j(y_j), \\ \quad \text{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1 \le j \le n} \nu_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=0} \min\limits_{1 \le j \le n} \nu_j(y_j), \\ \quad \text{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1 \le j \le n} \nu_j(y_j) \ge 0.5, \end{cases} \quad (A.58)$$

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \quad (i = 1, 2, \cdots, m), \quad (A.59)$$

$$\nu_j(y_j) = \begin{cases} b_j, & \text{if } y_j = 1 \\ 1 - b_j, & \text{if } y_j = 0 \end{cases} \quad (j = 1, 2, \cdots, n). \quad (A.60)$$

Proof: At first, when $(x_1, \cdots, x_m)$ is given, $f(x_1, \cdots, x_m, \tau_1, \cdots, \tau_n)$ is essentially a Boolean function of uncertain variables. It follows from the operational law of uncertain variables that

$$\mathcal{M}\{f(x_1, \cdots, x_m, \tau_1, \cdots, \tau_n) = 1\} = f^*(x_1, \cdots, x_m)$$

that is determined by (A.58). On the other hand, it follows from the operational law of uncertain random variables that

$$\text{Ch}\{\xi = 1\} = \sum_{(x_1, \cdots, x_m) \in \{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(x_i) \right) \mathcal{M}\{f(x_1, \cdots, x_m, \tau_1, \cdots, \tau_n) = 1\}.$$

Thus (A.57) is verified.

Remark A.6: When the uncertain variables disappear, the operational law becomes

$$\Pr\{\xi = 1\} = \sum_{(x_1, x_2, \cdots, x_m) \in \{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(x_i) \right) f(x_1, x_2, \cdots, x_m). \tag{A.61}$$

Remark A.7: When the random variables disappear, the operational law becomes

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \sup\limits_{f(y_1, y_2, \cdots, y_n) = 1} \min\limits_{1 \le j \le n} \nu_j(y_j), \\ \qquad \text{if } \sup\limits_{f(y_1, y_2, \cdots, y_n) = 1} \min\limits_{1 \le j \le n} \nu_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(y_1, y_2, \cdots, y_n) = 0} \min\limits_{1 \le j \le n} \nu_j(y_j), \\ \qquad \text{if } \sup\limits_{f(y_1, y_2, \cdots, y_n) = 1} \min\limits_{1 \le j \le n} \nu_j(y_j) \ge 0.5. \end{cases} \tag{A.62}$$

Exercise A.9: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent Boolean random variables defined by (A.54) and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent Boolean uncertain variables defined by (A.55). Then the minimum

$$\xi = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m \wedge \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n \tag{A.63}$$

is a Boolean uncertain random variable. Show that

$$\text{Ch}\{\xi = 1\} = a_1 a_2 \cdots a_m (b_1 \wedge b_2 \wedge \cdots \wedge b_n). \tag{A.64}$$

Exercise A.10: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent Boolean random variables defined by (A.54) and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent Boolean uncertain variables defined by (A.55). Then the maximum

$$\xi = \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n \tag{A.65}$$

is a Boolean uncertain random variable. Show that

$$\text{Ch}\{\xi = 1\} = 1 - (1 - a_1)(1 - a_2)\cdots(1 - a_m)(1 - b_1 \vee b_2 \vee \cdots \vee b_n). \quad \text{(A.66)}$$

Exercise A.11: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent Boolean random variables defined by (A.54) and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent Boolean uncertain variables defined by (A.55). Then the kth order statistic

$$\xi = k\text{-min}\,[\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n] \quad \text{(A.67)}$$

is a Boolean uncertain random variable. Show that

$$\text{Ch}\{\xi = 1\} = \sum_{(x_1,\cdots,x_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(x_i)\right) k\text{-min}\,[x_1, \cdots, x_m, b_1, \cdots, b_n]$$

where

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \quad (i = 1, 2, \cdots, m). \quad \text{(A.68)}$$

Exercise A.12: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent Boolean random variables defined by (A.54) and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent Boolean uncertain variables defined by (A.55). Then

$$\xi = k\text{-max}\,[\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n] \quad \text{(A.69)}$$

is the $(n - k + 1)$th order statistic. Show that

$$\text{Ch}\{\xi = 1\} = \sum_{(x_1,\cdots,x_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(x_i)\right) k\text{-max}\,[x_1, \cdots, x_m, b_1, \cdots, b_n]$$

where

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \quad (i = 1, 2, \cdots, m). \quad \text{(A.70)}$$

A.5 Expected Value

Definition A.5 *(Liu [105]) Let ξ be an uncertain random variable. Then its expected value is defined by*

$$E[\xi] = \int_0^{+\infty} \text{Ch}\{\xi \geq x\}\text{d}x - \int_{-\infty}^{0} \text{Ch}\{\xi \leq x\}\text{d}x \quad \text{(A.71)}$$

provided that at least one of the two integrals is finite.

Theorem A.16 *(Liu [105]) Let ξ be an uncertain random variable with chance distribution Φ. Then*

$$E[\xi]=\int_0^{+\infty}(1-\Phi(x))\mathrm{d}x-\int_{-\infty}^0\Phi(x)\mathrm{d}x. \tag{A.72}$$

Proof: It follows from the chance inversion theorem that for almost all numbers x, we have $\mathrm{Ch}\{\xi\geq x\}=1-\Phi(x)$ and $\mathrm{Ch}\{\xi\leq x\}=\Phi(x)$. By using the definition of expected value operator, we obtain

$$\begin{aligned}E[\xi] &= \int_0^{+\infty}\mathrm{Ch}\{\xi\geq x\}\mathrm{d}x-\int_{-\infty}^0\mathrm{Ch}\{\xi\leq x\}\mathrm{d}x\\ &= \int_0^{+\infty}(1-\Phi(x))\mathrm{d}x-\int_{-\infty}^0\Phi(x)\mathrm{d}x.\end{aligned}$$

Thus we obtain the equation (A.72).

Theorem A.17 *Let ξ be an uncertain random variable with chance distribution Φ. Then*

$$E[\xi]=\int_{-\infty}^{+\infty}x\mathrm{d}\Phi(x). \tag{A.73}$$

Proof: It follows from the change of variables of integral and Theorem A.16 that the expected value is

$$\begin{aligned}E[\xi] &= \int_0^{+\infty}(1-\Phi(x))\mathrm{d}x-\int_{-\infty}^0\Phi(x)\mathrm{d}x\\ &= \int_0^{+\infty}x\mathrm{d}\Phi(x)+\int_{-\infty}^0x\mathrm{d}\Phi(x)=\int_{-\infty}^{+\infty}x\mathrm{d}\Phi(x).\end{aligned}$$

The theorem is proved.

Theorem A.18 *Let ξ be an uncertain random variable with regular chance distribution Φ. Then*

$$E[\xi]=\int_0^1\Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{A.74}$$

Proof: It follows from the change of variables of integral and Theorem A.16 that the expected value is

$$\begin{aligned}E[\xi] &= \int_0^{+\infty}(1-\Phi(x))\mathrm{d}x-\int_{-\infty}^0\Phi(x)\mathrm{d}x\\ &= \int_{\Phi(0)}^1\Phi^{-1}(\alpha)\mathrm{d}\alpha+\int_0^{\Phi(0)}\Phi^{-1}(\alpha)\mathrm{d}\alpha=\int_0^1\Phi^{-1}(\alpha)\mathrm{d}\alpha.\end{aligned}$$

The theorem is proved.

Theorem A.19 *(Liu [106]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables, and let f be a measurable function. Then*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) \tag{A.75}$$

has an expected value

$$E[\xi] = \int_{\Re^m} E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)] \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \cdots \mathrm{d}\Psi_m(y_m)$$

where $E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)]$ is the expected value of the uncertain variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$ for any real numbers $y_1, y_2, \cdots, y_m$.

Proof: For simplicity, we only prove the case $m = n = 2$. Write the uncertainty distribution of $f(y_1, y_2, \tau_1, \tau_2)$ by $F(x; y_1, y_2)$ for any real numbers y_1 and y_2. Then

$$E[f(y_1, y_2, \tau_1, \tau_2)] = \int_0^{+\infty} (1 - F(x; y_1, y_2)) \mathrm{d}x - \int_{-\infty}^0 F(x; y_1, y_2) \mathrm{d}x.$$

On the other hand, the uncertain random variable $\xi = f(\eta_1, \eta_2, \tau_1, \tau_2)$ has a chance distribution

$$\Phi(x) = \int_{\Re^2} F(x; y_1, y_2) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2).$$

It follows from Theorem A.16 that

$$\begin{aligned}
E[\xi] &= \int_0^{+\infty} (1 - \Phi(x)) \mathrm{d}x - \int_{-\infty}^0 \Phi(x) \mathrm{d}x \\
&= \int_0^{+\infty} \left(1 - \int_{\Re^2} F(x; y_1, y_2) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2)\right) \mathrm{d}x \\
&\qquad - \int_{-\infty}^0 \int_{\Re^2} F(x; y_1, y_2) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \mathrm{d}x \\
&= \int_{\Re^2} \left(\int_0^{+\infty} (1 - F(x; y_1, y_2)) \mathrm{d}x - \int_{-\infty}^0 F(x; y_1, y_2) \mathrm{d}x\right) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&= \int_{\Re^2} E[f(y_1, y_2, \tau_1, \tau_2)] \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2).
\end{aligned}$$

Thus the theorem is proved.

Exercise A.13: Let η be a random variable and let τ be an uncertain variable. Show that

$$E[\eta + \tau] = E[\eta] + E[\tau] \tag{A.76}$$

and

$$E[\eta\tau] = E[\eta]E[\tau]. \tag{A.77}$$

Exercise A.14: Let η be a random variable with probability distribution Ψ, and let τ be an uncertain variable with regular uncertainty distribution Υ. Show that

$$E[\eta \vee \tau] = \int_{\Re}\int_0^1 \big(y \vee \Upsilon^{-1}(\alpha)\big)\,\mathrm{d}\alpha\mathrm{d}\Psi(y) \tag{A.78}$$

and

$$E[\eta \wedge \tau] = \int_{\Re}\int_0^1 \big(y \wedge \Upsilon^{-1}(\alpha)\big)\,\mathrm{d}\alpha\mathrm{d}\Psi(y). \tag{A.79}$$

Theorem A.20 *(Liu [106], Linearity of Expected Value Operator) Assume η_1 and η_2 are random variables (not necessarily independent), τ_1 and τ_2 are independent uncertain variables, and f_1 and f_2 are measurable functions. Then*

$$E[f_1(\eta_1,\tau_1) + f_2(\eta_2,\tau_2)] = E[f_1(\eta_1,\tau_1)] + E[f_2(\eta_2,\tau_2)]. \tag{A.80}$$

Proof: Since τ_1 and τ_2 are independent uncertain variables, for any real numbers y_1 and y_2, the functions $f_1(y_1,\tau_1)$ and $f_2(y_2,\tau_2)$ are also independent uncertain variables. Thus

$$E[f_1(y_1,\tau_1) + f_2(y_2,\tau_2)] = E[f_1(y_1,\tau_1)] + E[f_2(y_2,\tau_2)].$$

Let Ψ_1 and Ψ_2 be the probability distributions of random variables η_1 and η_2, respectively. Then we have

$$\begin{aligned}
&E[f_1(\eta_1,\tau_1) + f_2(\eta_2,\tau_2)]\\
&= \int_{\Re^2} E[f_1(y_1,\tau_1) + f_2(y_2,\tau_2)]\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\\
&= \int_{\Re^2} (E[f_1(y_1,\tau_1)] + E[f_2(y_2,\tau_2)])\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\\
&= \int_{\Re} E[f_1(y_1,\tau_1)]\mathrm{d}\Psi_1(y_1) + \int_{\Re} E[f_2(y_2,\tau_2)]\mathrm{d}\Psi_2(y_2)\\
&= E[f_1(\eta_1,\tau_1)] + E[f_2(\eta_2,\tau_2)].
\end{aligned}$$

The theorem is proved.

Exercise A.15: Assume η_1 and η_2 are random variables, and τ_1 and τ_2 are independent uncertain variables. Show that

$$E[\eta_1 \vee \tau_1 + \eta_2 \wedge \tau_2] = E[\eta_1 \vee \tau_1] + E[\eta_2 \wedge \tau_2]. \tag{A.81}$$

Theorem A.21 *(Liu [105]) Let ξ be an uncertain random variable, and let f be a nonnegative even function. If f is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$, then for any given number $t > 0$, we have*

$$\text{Ch}\{|\xi| \geq t\} \leq \frac{E[f(\xi)]}{f(t)}. \tag{A.82}$$

Proof: It is clear that $\text{Ch}\{|\xi| \geq f^{-1}(r)\}$ is a monotone decreasing function of r on $[0, \infty)$. It follows from the nonnegativity of $f(\xi)$ that

$$\begin{aligned} E[f(\xi)] &= \int_0^{+\infty} \text{Ch}\{f(\xi) \geq x\}\mathrm{d}x = \int_0^{+\infty} \text{Ch}\{|\xi| \geq f^{-1}(x)\}\mathrm{d}x \\ &\geq \int_0^{f(t)} \text{Ch}\{|\xi| \geq f^{-1}(x)\}\mathrm{d}x \geq \int_0^{f(t)} \text{Ch}\{|\xi| \geq f^{-1}(f(t))\}\mathrm{d}x \\ &= \int_0^{f(t)} \text{Ch}\{|\xi| \geq t\}\mathrm{d}x = f(t) \cdot \text{Ch}\{|\xi| \geq t\} \end{aligned}$$

which proves the inequality.

Theorem A.22 *(Liu [105], Markov Inequality) Let ξ be an uncertain random variable. Then for any given numbers $t > 0$ and $p > 0$, we have*

$$\text{Ch}\{|\xi| \geq t\} \leq \frac{E[|\xi|^p]}{t^p}. \tag{A.83}$$

Proof: It is a special case of Theorem A.21 when $f(x) = |x|^p$.

A.6 Variance

Definition A.6 *(Liu [105]) Let ξ be an uncertain random variable with finite expected value e. Then the variance of ξ is*

$$V[\xi] = E[(\xi - e)^2]. \tag{A.84}$$

Since $(\xi - e)^2$ is a nonnegative uncertain random variable, we also have

$$V[\xi] = \int_0^{+\infty} \text{Ch}\{(\xi - e)^2 \geq x\}\mathrm{d}x. \tag{A.85}$$

Theorem A.23 *(Liu [105]) If ξ is an uncertain random variable with finite expected value, a and b are real numbers, then*

$$V[a\xi + b] = a^2 V[\xi]. \tag{A.86}$$

Proof: Let e be the expected value of ξ. Then $a\xi + b$ has an expected value $ae + b$. Thus the variance is

$$V[a\xi + b] = E[(a\xi + b - (ae + b))^2] = E[a^2(\xi - e)^2] = a^2 V[\xi].$$

The theorem is verified.

Theorem A.24 *(Liu [105]) Let ξ be an uncertain random variable with expected value e. Then $V[\xi]=0$ if and only if* $\text{Ch}\{\xi=e\}=1$.

Proof: We first assume $V[\xi]=0$. It follows from the equation (A.85) that

$$\int_0^{+\infty}\text{Ch}\{(\xi-e)^2\ge x\}\mathrm{d}x=0$$

which implies $\text{Ch}\{(\xi-e)^2\ge x\}=0$ for any $x>0$. Hence we have

$$\text{Ch}\{(\xi-e)^2=0\}=1.$$

That is, $\text{Ch}\{\xi=e\}=1$. Conversely, assume $\text{Ch}\{\xi=e\}=1$. Then we immediately have $\text{Ch}\{(\xi-e)^2=0\}=1$ and $\text{Ch}\{(\xi-e)^2\ge x\}=0$ for any $x>0$. Thus

$$V[\xi]=\int_0^{+\infty}\text{Ch}\{(\xi-e)^2\ge x\}\mathrm{d}x=0.$$

The theorem is proved.

Theorem A.25 *(Liu [105], Chebyshev Inequality) Let ξ be an uncertain random variable whose variance exists. Then for any given number $t>0$, we have*

$$\text{Ch}\left\{|\xi-E[\xi]|\ge t\right\}\le\frac{V[\xi]}{t^2}. \tag{A.87}$$

Proof: It is a special case of Theorem A.21 when the uncertain random variable ξ is replaced with $\xi-E[\xi]$, and $f(x)=x^2$.

How to Obtain Variance from Distributions?

Let ξ be an uncertain random variable with expected value e. If we only know its chance distribution Φ, then the variance

$$\begin{aligned}
V[\xi]&=\int_0^{+\infty}\text{Ch}\{(\xi-e)^2\ge x\}\mathrm{d}x\\
&=\int_0^{+\infty}\text{Ch}\{(\xi\ge e+\sqrt{x})\cup(\xi\le e-\sqrt{x})\}\mathrm{d}x\\
&\le\int_0^{+\infty}(\text{Ch}\{\xi\ge e+\sqrt{x}\}+\text{Ch}\{\xi\le e-\sqrt{x}\})\mathrm{d}x\\
&=\int_0^{+\infty}(1-\Phi(e+\sqrt{x})+\Phi(e-\sqrt{x}))\mathrm{d}x.
\end{aligned}$$

Thus we have the following stipulation.

Stipulation A.1 *(Guo-Wang [51]) Let ξ be an uncertain random variable with chance distribution Φ and finite expected value e. Then*

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x. \tag{A.88}$$

Theorem A.26 *(Sheng-Yao [136]) Let ξ be an uncertain random variable with chance distribution Φ and finite expected value e. Then*

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x). \tag{A.89}$$

Proof: This theorem is based on Stipulation A.1 that says the variance of ξ is

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{y}))\mathrm{d}y + \int_0^{+\infty} \Phi(e - \sqrt{y})\mathrm{d}y.$$

Substituting $e + \sqrt{y}$ with x and y with $(x - e)^2$, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(e + \sqrt{y}))\mathrm{d}y = \int_e^{+\infty} (1 - \Phi(x))\mathrm{d}(x - e)^2 = \int_e^{+\infty} (x - e)^2 \mathrm{d}\Phi(x).$$

Similarly, substituting $e - \sqrt{y}$ with x and y with $(x - e)^2$, we obtain

$$\int_0^{+\infty} \Phi(e - \sqrt{y})\mathrm{d}y = \int_e^{-\infty} \Phi(x)\mathrm{d}(x - e)^2 = \int_{-\infty}^{e} (x - e)^2 \mathrm{d}\Phi(x).$$

It follows that the variance is

$$V[\xi] = \int_e^{+\infty} (x - e)^2 \mathrm{d}\Phi(x) + \int_{-\infty}^{e} (x - e)^2 \mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x).$$

The theorem is verified.

Theorem A.27 *(Sheng-Yao [136]) Let ξ be an uncertain random variable with regular chance distribution Φ and finite expected value e. Then*

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 \mathrm{d}\alpha. \tag{A.90}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem A.26 that the variance is

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 \mathrm{d}\alpha.$$

The theorem is verified.

Theorem A.28 *(Guo-Wang [51]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Assume $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \tau_{k+2}, \cdots, \tau_n$. Then*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) \tag{A.91}$$

has a variance

$$\begin{aligned} V[\xi] = &\int_{\Re^m}\int_0^{+\infty}(1 - F(e+\sqrt{x}; y_1, y_2, \cdots, y_m) \\ &+F(e-\sqrt{x}; y_1, y_2, \cdots, y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\cdots\Psi_m(y_m) \end{aligned}$$

where $F(x; y_1, y_2, \cdots, y_m)$ is the root α of the equation

$$f(y_1, y_2, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) = x.$$

Proof: It follows from the operational law of uncertain random variables that ξ has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, y_2, \cdots, y_m)\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\cdots\Psi_m(y_m)$$

where $F(x; y_1, y_2, \cdots, y_m)$ is the uncertainty distribution of the uncertain variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$. Thus the theorem follows Stipulation A.1 immediately.

Exercise A.16: Let η be a random variable with probability distribution Ψ, and let τ be an uncertain variable with uncertainty distribution Υ. Show that the sum

$$\xi = \eta + \tau \tag{A.92}$$

has a variance

$$V[\xi] = \int_{-\infty}^{+\infty}\int_0^{+\infty}(1 - \Upsilon(e+\sqrt{x}-y) + \Upsilon(e-\sqrt{x}-y))\mathrm{d}x\mathrm{d}\Psi(y). \tag{A.93}$$

A.7 Law of Large Numbers

Theorem A.29 *(Yao-Gao [179], Law of Large Numbers) Let $\eta_1, \eta_2, \cdots$ be iid random variables with a common probability distribution Ψ, and let $\tau_1, \tau_2, \cdots$ be iid uncertain variables. Assume f is a strictly monotone function. Then*

$$S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n) \tag{A.94}$$

is a sequence of uncertain random variables and

$$\frac{S_n}{n} \to \int_{-\infty}^{+\infty} f(y, \tau_1)\mathrm{d}\Psi(y) \tag{A.95}$$

in the sense of convergence in distribution as $n \to \infty$.

Proof: According to the definition of convergence in distribution, it suffices to prove

$$\begin{aligned} &\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le \int_{-\infty}^{+\infty} f(y, z)\mathrm{d}\Psi(y)\right\} \\ &= \mathcal{M}\left\{\int_{-\infty}^{+\infty} f(y, \tau_1)\mathrm{d}\Psi(y) \le \int_{-\infty}^{+\infty} f(y, z)\mathrm{d}\Psi(y)\right\} \end{aligned} \tag{A.96}$$

for any real number z with

$$\begin{aligned} &\lim_{w\to z} \mathcal{M}\left\{\int_{-\infty}^{+\infty} f(y, \tau_1)\mathrm{d}\Psi(y) \le \int_{-\infty}^{+\infty} f(y, w)\mathrm{d}\Psi(y)\right\} \\ &= \mathcal{M}\left\{\int_{-\infty}^{+\infty} f(y, \tau_1)\mathrm{d}\Psi(y) \le \int_{-\infty}^{+\infty} f(y, z)\mathrm{d}\Psi(y)\right\}. \end{aligned}$$

The argument breaks into two cases. Case 1: Assume $f(y, z)$ is strictly increasing with respect to z. Let Υ denote the common uncertainty distribution of $\tau_1, \tau_2, \cdots$ It is clear that

$$\mathcal{M}\{f(y, \tau_1) \le f(y, z)\} = \Upsilon(z)$$

for any real numbers y and z. Thus we have

$$\mathcal{M}\left\{\int_{-\infty}^{+\infty} f(y, \tau_1)\mathrm{d}\Psi(y) \le \int_{-\infty}^{+\infty} f(y, z)\mathrm{d}\Psi(y)\right\} = \Upsilon(z). \tag{A.97}$$

In addition, since $f(\eta_1, z), f(\eta_2, z), \cdots$ are a sequence of iid random variables, the law of large numbers for random variables tells us that

$$\frac{f(\eta_1, z) + f(\eta_2, z) + \cdots + f(\eta_n, z)}{n} \to \int_{-\infty}^{+\infty} f(y, z)\mathrm{d}\Psi(y), \quad a.s.$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le \int_{-\infty}^{+\infty} f(y, z)\mathrm{d}\Psi(y)\right\} = \Upsilon(z). \tag{A.98}$$

It follows from (A.97) and (A.98) that (A.96) holds. Case 2: Assume $f(y, z)$ is strictly decreasing with respect to z. Then $-f(y, z)$ is strictly increasing with respect to z. By using Case 1 we obtain

$$\lim_{n\to\infty} \mathrm{Ch}\left\{-\frac{S_n}{n} < -z\right\} = \mathcal{M}\left\{-\int_{-\infty}^{+\infty} f(y, \tau_1)\mathrm{d}\Psi(y) < -z\right\}.$$

That is,

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} > z\right\} = \mathcal{M}\left\{\int_{-\infty}^{+\infty} f(y,\tau_1)\mathrm{d}\Psi(y) > z\right\}.$$

It follows from the duality property that

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\frac{S_n}{n} \le z\right\} = \mathcal{M}\left\{\int_{-\infty}^{+\infty} f(y,\tau_1)\mathrm{d}\Psi(y) \le z\right\}.$$

The theorem is thus proved.

Exercise A.17: Let $\eta_1, \eta_2, \cdots$ be iid random variables, and let $\tau_1, \tau_2, \cdots$ be iid uncertain variables. Define

$$S_n = (\eta_1 + \tau_1) + (\eta_2 + \tau_2) + \cdots + (\eta_n + \tau_n). \tag{A.99}$$

Show that

$$\frac{S_n}{n} \to E[\eta_1] + \tau_1 \tag{A.100}$$

in the sense of convergence in distribution as $n \to \infty$.

Exercise A.18: Let $\eta_1, \eta_2, \cdots$ be iid positive random variables, and let $\tau_1, \tau_2, \cdots$ be iid positive uncertain variables. Define

$$S_n = \eta_1\tau_1 + \eta_2\tau_2 + \cdots + \eta_n\tau_n. \tag{A.101}$$

Show that

$$\frac{S_n}{n} \to E[\eta_1]\tau_1 \tag{A.102}$$

in the sense of convergence in distribution as $n \to \infty$.

A.8 Uncertain Random Programming

Assume that $\boldsymbol{x}$ is a decision vector, and $\boldsymbol{\xi}$ is an uncertain random vector. Since an uncertain random objective function $f(\boldsymbol{x}, \boldsymbol{\xi})$ cannot be directly minimized, we may minimize its expected value, i.e.,

$$\min_{\boldsymbol{x}} E[f(\boldsymbol{x}, \boldsymbol{\xi})]. \tag{A.103}$$

Since the uncertain random constraints $g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0, j = 1, 2, \cdots, p$ do not make a crisp feasible set, it is naturally desired that the uncertain random constraints hold with confidence levels $\alpha_1, \alpha_2, \cdots, \alpha_p$. Then we have a set of chance constraints,

$$\mathrm{Ch}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1, 2, \cdots, p. \tag{A.104}$$

In order to obtain a decision with minimum expected objective value subject to a set of chance constraints, Liu [106] proposed the following uncertain random programming model,

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[f(\boldsymbol{x}, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \text{Ch}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1, 2, \cdots, p. \end{cases} \tag{A.105}$$

Definition A.7 *(Liu [106]) A vector $\boldsymbol{x}$ is called a feasible solution to the uncertain random programming model (A.105) if*

$$\text{Ch}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j \tag{A.106}$$

for $j = 1, 2, \cdots, p$.

Definition A.8 *(Liu [106]) A feasible solution $\boldsymbol{x}^*$ is called an optimal solution to the uncertain random programming model (A.105) if*

$$E[f(\boldsymbol{x}^*, \boldsymbol{\xi})] \le E[f(\boldsymbol{x}, \boldsymbol{\xi})] \tag{A.107}$$

for any feasible solution $\boldsymbol{x}$.

Theorem A.30 *(Liu [106]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is a strictly increasing function or a strictly decreasing function with respect to $\tau_1, \cdots, \tau_n$, then the expected function*

$$E[f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)] \tag{A.108}$$

is equal to

$$\int_{\Re^m} \int_0^1 f(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m).$$

Proof: Since $f(\boldsymbol{x}, y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)$ is a strictly increasing function or a strictly decreasing function with respect to $\tau_1, \cdots, \tau_n$, we have

$$E[f(\boldsymbol{x}, y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)] = \int_0^1 f(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) \mathrm{d}\alpha.$$

It follows from Theorem A.19 that the result holds.

Remark A.8: If $f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, then the integrand in the formula of expected value $E[f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)]$ should be replaced with

$$f(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)).$$

Theorem A.31 *(Liu [106]) Let* $\eta_1, \eta_2, \cdots, \eta_m$ *be independent random variables with probability distributions* $\Psi_1, \Psi_2, \cdots, \Psi_m$*, and let* $\tau_1, \tau_2, \cdots, \tau_n$ *be independent uncertain variables with regular uncertainty distributions* $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$*, respectively. If* $g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ *is a strictly increasing function with respect to* $\tau_1, \cdots, \tau_n$*, then the chance constraint*

$$\mathrm{Ch}\{g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \le 0\} \ge \alpha_j \tag{A.109}$$

holds if and only if

$$\int_{\Re^m} G_j(\boldsymbol{x}, y_1, \cdots, y_m)\mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \ge \alpha_j \tag{A.110}$$

where $G_j(\boldsymbol{x}, y_1, \cdots, y_m)$ *is the root* α *of the equation*

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) = 0. \tag{A.111}$$

Proof: It follows from Theorem A.6 that the left side of the chance constraint (A.109) is

$$\begin{aligned}
&\mathrm{Ch}\{g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \le 0\} \\
&= \int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{g_j(\boldsymbol{x}, \eta_1(\omega), \cdots, \eta_m(\omega), \tau_1, \cdots, \tau_n) \le 0\} \ge r\}\, \mathrm{d}r \\
&= \int_{\Re^m} \mathcal{M}\{g_j(\boldsymbol{x}, y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) \le 0\}\mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \\
&= \int_{\Re^m} G_j(\boldsymbol{x}; y_1, \cdots, y_m)\mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m)
\end{aligned}$$

where $G_j(\boldsymbol{x}, y_1, \cdots, y_m) = \mathcal{M}\{g_j(\boldsymbol{x}, y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) \le 0\}$ is the root α of the equation (A.111). Hence the chance constraint (A.109) holds if and only if (A.110) is true. The theorem is verified.

Remark A.9: Sometimes, the equation (A.111) may not have a root. In this case, if

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) < 0 \tag{A.112}$$

for all α, then we set the root $\alpha = 1$; and if

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) > 0 \tag{A.113}$$

for all α, then we set the root $\alpha = 0$.

Remark A.10: The root α may be estimated by the bisection method because $g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha))$ is a strictly increasing function with respect to α.

Remark A.11: If $g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, then the equation (A.111) becomes

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) = 0.$$

Theorem A.32 *(Liu [106]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the objective function $f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ and constraint functions $g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ are strictly increasing functions with respect to $\tau_1, \cdots, \tau_n$ for $j = 1, 2, \cdots, p$, then the uncertain random programming*

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)] \\ \textit{subject to:} \\ \qquad \mathrm{Ch}\{g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \le 0\} \ge \alpha_j, \ j = 1, 2, \cdots, p \end{cases}$$

is equivalent to the crisp mathematical programming

$$\begin{cases} \min\limits_{\boldsymbol{x}} \displaystyle\int_{\Re^m} \int_0^1 f(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \\ \textit{subject to:} \\ \qquad \displaystyle\int_{\Re^m} G_j(\boldsymbol{x}, y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \ge \alpha_j, \ j = 1, 2, \cdots, p \end{cases}$$

where $G_j(\boldsymbol{x}, y_1, \cdots, y_m)$ are the roots α of the equations

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) = 0 \tag{A.114}$$

for $j = 1, 2, \cdots, p$, respectively.

Proof: It follows from Theorems A.30 and A.31 immediately.

After an uncertain random programming is converted into a crisp mathematical programming, we may solve it by any classical numerical methods (e.g. iterative method) or intelligent algorithms (e.g. genetic algorithm).

A.9 Uncertain Random Risk Analysis

The study of uncertain random risk analysis was started by Liu-Ralescu [107] with the concept of risk index.

Definition A.9 *(Liu-Ralescu [107]) Assume that a system contains uncertain random factors $\xi_1, \xi_2, \cdots, \xi_n$, and has a loss function f. Then the risk index is the chance measure that the system is loss-positive, i.e.,*

$$\mathit{Risk} = \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{A.115}$$

If all uncertain random factors degenerate to random ones, then the risk index is the probability measure that the system is loss-positive (Roy [129]). If all uncertain random factors degenerate to uncertain ones, then the risk index is the uncertain measure that the system is loss-positive (Liu [82]).

Theorem A.33 *Assume that a system contains uncertain random factors* $\xi_1, \xi_2, \cdots, \xi_n$*, and has a loss function* f*. If* $f(\xi_1, \xi_2, \cdots, \xi_n)$ *has a chance distribution* Φ*, then the risk index is*

$$Risk = 1 - \Phi(0). \tag{A.116}$$

Proof: It follows from the definition of risk index and self-duality of chance measure that

$$\begin{aligned} Risk &= \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\} \\ &= 1 - \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) \le 0\} \\ &= 1 - \Phi(0). \end{aligned}$$

The theorem is proved.

Theorem A.34 *(Liu-Ralescu [107], Risk Index Theorem) Assume a system contains independent random variables* $\eta_1, \eta_2, \cdots, \eta_m$ *with probability distributions* $\Psi_1, \Psi_2, \cdots, \Psi_m$ *and independent uncertain variables* $\tau_1, \tau_2, \cdots, \tau_n$ *with regular uncertainty distributions* $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$*, respectively. If the loss function* $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ *is strictly increasing with respect to* $\tau_1, \cdots, \tau_k$ *and strictly decreasing with respect to* $\tau_{k+1}, \cdots, \tau_n$*, then the risk index is*

$$Risk = \int_{\Re^m} G(y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \tag{A.117}$$

where $G(y_1, \cdots, y_m)$ *is the root* α *of the equation*

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(1-\alpha), \cdots, \Upsilon_k^{-1}(1-\alpha), \Upsilon_{k+1}^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) = 0.$$

Proof: It follows from the definition of risk index and Theorem A.6 that

$$\begin{aligned} Risk &= \mathrm{Ch}\{f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) > 0\} \\ &= \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{f(\eta_1(\omega), \cdots, \eta_m(\omega), \tau_1, \cdots, \tau_n) > 0\} \ge r\} \mathrm{d}r \\ &= \int_{\Re^m} \mathcal{M}\{f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) > 0\} \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \\ &= \int_{\Re^m} G(y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \end{aligned}$$

where $G(y_1, \cdots, y_m) = \mathcal{M}\{f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) > 0\}$ is the root α of the equation

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(1-\alpha), \cdots, \Upsilon_k^{-1}(1-\alpha), \Upsilon_{k+1}^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) = 0.$$

The theorem is thus verified.

Remark A.12: Sometimes, the equation may not have a root. In this case, if

$$f(y_1,\cdots,y_m,\Upsilon_1^{-1}(1-\alpha),\cdots,\Upsilon_k^{-1}(1-\alpha),\Upsilon_{k+1}^{-1}(\alpha),\cdots,\Upsilon_n^{-1}(\alpha))<0$$

for all α, then we set the root $\alpha=0$; and if

$$f(y_1,\cdots,y_m,\Upsilon_1^{-1}(1-\alpha),\cdots,\Upsilon_k^{-1}(1-\alpha),\Upsilon_{k+1}^{-1}(\alpha),\cdots,\Upsilon_n^{-1}(\alpha))>0$$

for all α, then we set the root $\alpha=1$.

Remark A.13: The root α may be estimated by the bisection method because $f(y_1,\cdots,y_m,\Upsilon_1^{-1}(1-\alpha),\cdots,\Upsilon_k^{-1}(1-\alpha),\Upsilon_{k+1}^{-1}(\alpha),\cdots,\Upsilon_n^{-1}(\alpha))$ is a strictly decreasing function with respect to α.

Exercise A.19: (Series System) Consider a series system in which there are m elements whose lifetimes are independent random variables $\eta_1,\eta_2,\cdots,\eta_m$ with continuous probability distributions $\Psi_1,\Psi_2,\cdots,\Psi_m$ and n elements whose lifetimes are independent uncertain variables $\tau_1,\tau_2,\cdots,\tau_n$ with continuous uncertainty distributions $\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f=T-\eta_1\wedge\eta_2\wedge\cdots\wedge\eta_m\wedge\tau_1\wedge\tau_2\wedge\cdots\wedge\tau_n. \tag{A.118}$$

Show that the risk index is

$$Risk=a+b-ab \tag{A.119}$$

where

$$a=1-(1-\Psi_1(T))(1-\Psi_2(T))\cdots(1-\Psi_m(T)), \tag{A.120}$$

$$b=\Upsilon_1(T)\vee\Upsilon_2(T)\vee\cdots\vee\Upsilon_n(T). \tag{A.121}$$

Exercise A.20: (Parallel System) Consider a parallel system in which there are m elements whose lifetimes are independent random variables $\eta_1,\eta_2,\cdots,\eta_m$ with continuous probability distributions $\Psi_1,\Psi_2,\cdots,\Psi_m$ and n elements whose lifetimes are independent uncertain variables $\tau_1,\tau_2,\cdots,\tau_n$ with continuous uncertainty distributions $\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f=T-\eta_1\vee\eta_2\vee\cdots\vee\eta_m\vee\tau_1\vee\tau_2\vee\cdots\vee\tau_n. \tag{A.122}$$

Show that the risk index is

$$Risk=ab \tag{A.123}$$

where

$$a = \Psi_1(T)\Psi_2(T)\cdots\Psi_m(T), \tag{A.124}$$

$$b = \Upsilon_1(T) \wedge \Upsilon_2(T) \wedge \cdots \wedge \Upsilon_n(T). \tag{A.125}$$

Exercise A.21: (k-out-of-($m+n$) System) Consider a k-out-of-($m+n$) system in which there are m elements whose lifetimes are independent random variables $\eta_1, \eta_2, \cdots, \eta_m$ with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$ and n elements whose lifetimes are independent uncertain variables $\tau_1, \tau_2, \cdots, \tau_n$ with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f = T - k\text{-max}[\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n]. \tag{A.126}$$

Show that the risk index is

$$Risk = \int_{\Re^m} G(y_1, y_2, \cdots, y_m) \mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\cdots\mathrm{d}\Psi_m(y_m) \tag{A.127}$$

where $G(y_1, y_2, \cdots, y_m)$ is the root α of the equation

$$k\text{-max}[y_1, y_2, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \Upsilon_2^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)] = T. \tag{A.128}$$

Exercise A.22: (Standby System) Consider a standby system in which there are m elements whose lifetimes are independent random variables $\eta_1, \eta_2, \cdots, \eta_m$ with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$ and n elements whose lifetimes are independent uncertain variables $\tau_1, \tau_2, \cdots, \tau_n$ with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f = T - (\eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n). \tag{A.129}$$

Show that the risk index is

$$Risk = \int_{\Re^m} G(y_1, y_2, \cdots, y_m) \mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\cdots\mathrm{d}\Psi_m(y_m) \tag{A.130}$$

where $G(y_1, y_2, \cdots, y_m)$ is the root α of the equation

$$\Upsilon_1^{-1}(\alpha) + \Upsilon_2^{-1}(\alpha) + \cdots + \Upsilon_n^{-1}(\alpha) = T - (y_1 + y_2 + \cdots + y_m). \tag{A.131}$$

Remark A.14: As a substitute of risk index, Liu-Ralescu [109] suggested a concept of value-at-risk,

$$\mathrm{VaR}(\alpha) = \sup\{x \mid \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) \ge x\} \ge \alpha\}. \tag{A.132}$$

Note that $\mathrm{VaR}(\alpha)$ represents the maximum possible loss when α percent of the right tail distribution is ignored. In other words, the loss will exceed $\mathrm{VaR}(\alpha)$ with chance measure α. If the chance distribution $\Phi(x)$ of $f(\xi_1, \xi_2, \cdots, \xi_n)$ is continuous, then

$$\mathrm{VaR}(\alpha) = \sup\{x \mid \Phi(x) \leq 1 - \alpha\}. \tag{A.133}$$

If its inverse chance distribution $\Phi^{-1}(\alpha)$ exists, then

$$\mathrm{VaR}(\alpha) = \Phi^{-1}(1 - \alpha). \tag{A.134}$$

It is also easy to show that $\mathrm{VaR}(\alpha)$ is a monotone decreasing function with respect to α. When the uncertain random variables degenerate to random variables, the value-at-risk becomes the one in Morgan [114]. When the uncertain random variables degenerate to uncertain variables, the value-at-risk becomes the one in Peng [119].

Remark A.15: Liu-Ralescu [111] proposed a concept of expected loss that is the expected value of the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ given $f(\xi_1, \xi_2, \cdots, \xi_n) > 0$, i.e.,

$$L = \int_0^{+\infty} \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq x\}\mathrm{d}x. \tag{A.135}$$

If $\Phi(x)$ is the chance distribution of the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$, then we immediately have

$$L = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x. \tag{A.136}$$

If its inverse chance distribution $\Phi^{-1}(\alpha)$ exists, then the expected loss is

$$L = \int_0^1 \left(\Phi^{-1}(\alpha)\right)^+ \mathrm{d}\alpha. \tag{A.137}$$

A.10 Uncertain Random Reliability Analysis

The study of uncertain random reliability analysis was started by Wen-Kang [153] with the concept of reliability index.

Definition A.10 *(Wen-Kang [153]) Assume a Boolean system has uncertain random elements $\xi_1, \xi_2, \cdots, \xi_n$ and a structure function f. Then the reliability index is the chance measure that the system is working, i.e.,*

$$\textit{Reliability} = \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) = 1\}. \tag{A.138}$$

If all uncertain random elements degenerate to random ones, then the reliability index is the probability measure that the system is working. If all uncertain random elements degenerate to uncertain ones, then the reliability index (Liu [82]) is the uncertain measure that the system is working.

Theorem A.35 *(Wen-Kang [153], Reliability Index Theorem) Assume that a system has a structure function f and contains independent random elements $\eta_1, \eta_2, \cdots, \eta_m$ with reliabilities $a_1, a_2, \cdots, a_m$, and independent uncertain elements $\tau_1, \tau_2, \cdots, \tau_n$ with reliabilities $b_1, b_2, \cdots, b_n$, respectively. Then the reliability index is*

$$\text{Reliability} = \sum_{(x_1,\cdots,x_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(x_i)\right) f^*(x_1, \cdots, x_m) \tag{A.139}$$

where

$$f^*(x_1,\cdots,x_m) = \begin{cases} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \text{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=0} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \text{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) \ge 0.5, \end{cases} \tag{A.140}$$

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \qquad (i = 1, 2, \cdots, m), \tag{A.141}$$

$$\nu_j(y_j) = \begin{cases} b_j, & \text{if } y_j = 1 \\ 1 - b_j, & \text{if } y_j = 0 \end{cases} \qquad (j = 1, 2, \cdots, n). \tag{A.142}$$

Proof: It follows from Definition A.10 and Theorem A.15 immediately.

Exercise A.23: (Series System) Consider a series system in which there are m independent random elements $\eta_1, \eta_2, \cdots, \eta_m$ with reliabilities $a_1, a_2, \cdots, a_m$, and n independent uncertain elements $\tau_1, \tau_2, \cdots, \tau_n$ with reliabilities $b_1, b_2, \cdots$, b_n, respectively. Note that the structure function is

$$f = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m \wedge \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n. \tag{A.143}$$

Show that the reliability index is

$$\text{Reliability} = a_1 a_2 \cdots a_m (b_1 \wedge b_2 \wedge \cdots \wedge b_n). \tag{A.144}$$

Exercise A.24: (Parallel System) Consider a parallel system in which there are m independent random elements $\eta_1, \eta_2, \cdots, \eta_m$ with reliabilities $a_1, a_2, \cdots, a_m$, and n independent uncertain elements $\tau_1, \tau_2, \cdots, \tau_n$ with reliabilities $b_1, b_2, \cdots, b_n$, respectively. Note that the structure function is

$$f = \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n. \tag{A.145}$$

Show that the reliability index is

$$Reliability = 1-(1-a_1)(1-a_2)\cdots(1-a_m)(1-b_1\vee b_2\vee\cdots\vee b_n). \quad \text{(A.146)}$$

Exercise A.25: (k-out-of-($m+n$) System) Consider a k-out-of-($m+n$) system in which there are m independent random elements $\eta_1,\eta_2,\cdots,\eta_m$ with reliabilities $a_1,a_2,\cdots,a_m$, and n independent uncertain elements $\tau_1,\tau_2,\cdots,\tau_n$ with reliabilities $b_1,b_2,\cdots,b_n$, respectively. Note that the structure function is

$$f = k\text{-max}\,[\eta_1,\eta_2,\cdots,\eta_m,\tau_1,\tau_2,\cdots,\tau_n]. \quad \text{(A.147)}$$

Show that the reliability index is

$$Reliability = \sum_{(x_1,\cdots,x_m)\in\{0,1\}^m}\left(\prod_{i=1}^m \mu_i(x_i)\right) k\text{-max}\,[x_1,\cdots,x_m,b_1,\cdots,b_n]$$

where

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1-a_i, & \text{if } x_i = 0 \end{cases} \qquad (i=1,2,\cdots,m). \quad \text{(A.148)}$$

A.11 Uncertain Random Graph

In classic graph theory, the edges and vertices are all deterministic, either exist or not. However, in practical applications, some indeterminate factors will no doubt appear in graphs. Thus it is reasonable to assume that in a graph some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure. In order to model this type of graph, Liu [92] presented a concept of uncertain random graph.

We say a graph is of order n if it has n vertices labeled by $1,2,\cdots,n$. In this section, we assume the graph is always of order n, and has a collection of vertices,

$$\mathcal{V} = \{1,2,\cdots,n\}. \quad \text{(A.149)}$$

Let us define two collections of edges,

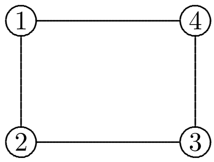
$$\mathcal{U} = \{(i,j)\,|\,1\le i<j\le n \text{ and } (i,j) \text{ are uncertain edges}\}, \quad \text{(A.150)}$$

$$\mathcal{R} = \{(i,j)\,|\,1\le i<j\le n \text{ and } (i,j) \text{ are random edges}\}. \quad \text{(A.151)}$$

Note that all deterministic edges are regarded as special uncertain ones. Then $\mathcal{U}\cup\mathcal{R} = \{(i,j)\,|\,1\le i<j\le n\}$ that contains $n(n-1)/2$ edges. We will call

$$\mathcal{T} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} \quad \text{(A.152)}$$

an *uncertain random adjacency matrix* if α_{ij} represent the truth values in uncertain measure or probability measure that the edges between vertices i and j exist, $i, j = 1, 2, \cdots, n$, respectively. Note that $\alpha_{ii} = 0$ for $i = 1, 2, \cdots, n$, and $\mathcal{T}$ is a symmetric matrix, i.e., $\alpha_{ij} = \alpha_{ji}$ for $i, j = 1, 2, \cdots, n$.



$$\begin{pmatrix} 0 & 0.8 & 0 & 0.5 \\ 0.8 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.3 \\ 0.5 & 0 & 0.3 & 0 \end{pmatrix}$$

Figure A.2: An Uncertain Random Graph

Definition A.11 *(Liu [92]) Assume* $\mathcal{V}$ *is the collection of vertices,* $\mathcal{U}$ *is the collection of uncertain edges,* $\mathcal{R}$ *is the collection of random edges, and* $\mathcal{T}$ *is the uncertain random adjacency matrix. Then the quartette* $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ *is said to be an uncertain random graph.*

Please note that the uncertain random graph becomes a *random graph* (Erdős-Rényi [28], Gilbert [50]) if the collection $\mathcal{U}$ of uncertain edges vanishes; and becomes an *uncertain graph* (Gao-Gao [42]) if the collection $\mathcal{R}$ of random edges vanishes.

In order to deal with uncertain random graph, let us introduce some symbols. Write

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \tag{A.153}$$

and

$$\mathbb{X} = \left\{ X \;\middle|\; \begin{array}{l} x_{ij} = 0 \text{ or } 1, \ \text{ if } (i,j) \in \mathcal{R} \\ x_{ij} = 0, \ \text{ if } (i,j) \in \mathcal{U} \\ x_{ij} = x_{ji}, \ i, j = 1, 2, \cdots, n \\ x_{ii} = 0, \ i = 1, 2, \cdots, n \end{array} \right\}. \tag{A.154}$$

For each given matrix

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}, \tag{A.155}$$

the *extension class* of Y is defined by

$$Y^* = \left\{ X \mid \begin{array}{c} x_{ij} = y_{ij}, \ \text{ if } (i,j) \in \mathcal{R} \\ x_{ij} = 0 \text{ or } 1, \ \text{ if } (i,j) \in \mathcal{U} \\ x_{ij} = x_{ji}, \ i,j = 1,2,\cdots,n \\ x_{ii} = 0, \ i = 1,2,\cdots,n \end{array} \right\}. \tag{A.156}$$

Example A.5: (Liu [92], Connectivity Index) An uncertain random graph is connected for some realizations of uncertain and random edges, and disconnected for some other realizations. In order to show how likely an uncertain random graph is connected, a connectivity index of an uncertain random graph is defined as the chance measure that the uncertain random graph is connected. Let $(\mathcal{V},\mathcal{U},\mathcal{R},\mathcal{T})$ be an uncertain random graph. Liu [92] proved that the connectivity index is

$$\rho = \sum_{Y\in\mathbb{X}} \left(\prod_{(i,j)\in\mathcal{R}} \nu_{ij}(Y) \right) f^*(Y) \tag{A.157}$$

where

$$f^*(Y) = \begin{cases} \displaystyle\sup_{X\in Y^*, f(X)=1} \min_{(i,j)\in\mathcal{U}} \nu_{ij}(X), & \text{if } \displaystyle\sup_{X\in Y^*, f(X)=1} \min_{(i,j)\in\mathcal{U}} \nu_{ij}(X) < 0.5 \\ 1 - \displaystyle\sup_{X\in Y^*, f(X)=0} \min_{(i,j)\in\mathcal{U}} \nu_{ij}(X), & \text{if } \displaystyle\sup_{X\in Y^*, f(X)=1} \min_{(i,j)\in\mathcal{U}} \nu_{ij}(X) \ge 0.5, \end{cases}$$

$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1-\alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases} \qquad (i,j) \in \mathcal{U}, \tag{A.158}$$

$$f(X) = \begin{cases} 1, & \text{if } I + X + X^2 + \cdots + X^{n-1} > 0 \\ 0, & \text{otherwise,} \end{cases} \tag{A.159}$$

$\mathbb{X}$ and Y^* are defined by (A.154) and (A.156), respectively.

Remark A.16: If the uncertain random graph becomes a random graph, then the connectivity index is

$$\rho = \sum_{X\in\mathbb{X}} \left(\prod_{1\le i<j\le n} \nu_{ij}(X) \right) f(X) \tag{A.160}$$

where

$$\mathbb{X} = \left\{ X \mid \begin{array}{c} x_{ij} = 0 \text{ or } 1, \ i,j = 1,2,\cdots,n \\ x_{ij} = x_{ji}, \ i,j = 1,2,\cdots,n \\ x_{ii} = 0, \ i = 1,2,\cdots,n \end{array} \right\}. \tag{A.161}$$

Remark A.17: (Gao-Gao [42]) If the uncertain random graph becomes an uncertain graph, then the connectivity index is

$$\rho = \begin{cases} \sup\limits_{X\in\mathbb{X}, f(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X), & \text{if } \sup\limits_{X\in\mathbb{X}, f(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{X\in\mathbb{X}, f(X)=0} \min\limits_{1\le i<j\le n} \nu_{ij}(X), & \text{if } \sup\limits_{X\in\mathbb{X}, f(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X) \ge 0.5 \end{cases}$$

where $\mathbb{X}$ becomes

$$\mathbb{X} = \left\{ X \;\middle|\; \begin{array}{c} x_{ij} = 0 \text{ or } 1,\ i,j = 1,2,\cdots,n \\ x_{ij} = x_{ji},\ i,j = 1,2,\cdots,n \\ x_{ii} = 0,\ i = 1,2,\cdots,n \end{array} \right\}. \tag{A.162}$$

Exercise A.26: (Zhang-Peng-Li [194]) An Euler circuit in the graph is a circuit that passes through each edge exactly once. In other words, a graph has an Euler circuit if it can be drawn on paper without ever lifting the pencil and without retracing over any edge. It has been proved that a graph has an Euler circuit if and only if it is connected and each vertex has an even degree (i.e., the number of edges that are adjacent to that vertex). In order to measure how likely an uncertain random graph has an Euler circuit, an Euler index is defined as the chance measure that the uncertain random graph has an Euler circuit. Please give a formula for calculating Euler index.

A.12 Uncertain Random Network

The term *network* is a synonym for a weighted graph, where the weights may be understood as cost, distance or time consumed. Assume that in a network some weights are random variables and others are uncertain variables. In order to model this type of network, Liu [92] presented a concept of uncertain random network.

In this section, we assume the uncertain random network is always of order n, and has a collection of nodes,

$$\mathcal{N} = \{1, 2, \cdots, n\} \tag{A.163}$$

where "1" is always the source node, and "n" is always the destination node. Let us define two collections of arcs,

$$\mathcal{U} = \{(i,j) \mid (i,j) \text{ are uncertain arcs}\}, \tag{A.164}$$

$$\mathcal{R} = \{(i,j) \mid (i,j) \text{ are random arcs}\}. \tag{A.165}$$

Note that all deterministic arcs are regarded as special uncertain ones. Let w_{ij} denote the weights of arcs (i,j), $(i,j) \in \mathcal{U} \cup \mathcal{R}$, respectively. Then w_{ij} are uncertain variables if $(i,j) \in \mathcal{U}$, and random variables if $(i,j) \in \mathcal{R}$. Write

$$\mathcal{W} = \{w_{ij} \mid (i,j) \in \mathcal{U} \cup \mathcal{R}\}. \tag{A.166}$$

Definition A.12 *(Liu [92]) Assume* $\mathcal{N}$ *is the collection of nodes,* $\mathcal{U}$ *is the collection of uncertain arcs,* $\mathcal{R}$ *is the collection of random arcs, and* $\mathcal{W}$ *is the collection of uncertain and random weights. Then the quartette* $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ *is said to be an uncertain random network.*

Please note that the uncertain random network becomes a *random network* (Frank-Hakimi [29]) if all weights are random variables; and becomes an *uncertain network* (Liu [83]) if all weights are uncertain variables.

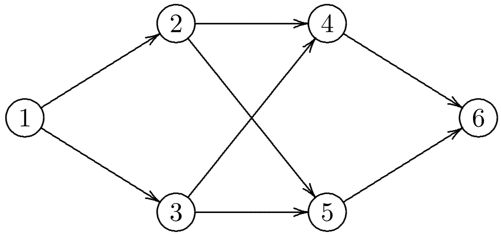


Figure A.3: An Uncertain Random Network

Figure A.3 shows an uncertain random network $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ of order 6 in which

$$\mathcal{N} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}, \tag{A.167}$$

$$\mathcal{U} = \{(1,2),\ (1,3),\ (2,4),\ (2,5),\ (3,4),\ (3,5)\}, \tag{A.168}$$

$$\mathcal{R} = \{(4,6),\ (5,6)\}, \tag{A.169}$$

$$\mathcal{W} = \{w_{12},\ w_{13},\ w_{24},\ w_{25},\ w_{34},\ w_{35},\ w_{46},\ w_{56}\}. \tag{A.170}$$

Example A.6: (Liu [92], Shortest Path Distribution) Consider an uncertain random network $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{W})$. Assume the uncertain weights w_{ij} have regular uncertainty distributions Υ_{ij} for $(i,j) \in \mathcal{U}$, and the random weights w_{ij} have probability distributions Ψ_{ij} for $(i,j) \in \mathcal{R}$, respectively. Then the shortest path distribution from a source node to a destination node is

$$\Phi(x) = \int_0^{+\infty} \cdots \int_0^{+\infty} F(x; y_{ij}, (i,j) \in \mathcal{R}) \prod_{(i,j) \in \mathcal{R}} \mathrm{d}\Psi_{ij}(y_{ij}) \tag{A.171}$$

where $F(x; y_{ij}, (i,j) \in \mathcal{R})$ is the root α of the equation

$$f(\Upsilon_{ij}^{-1}(\alpha), (i,j) \in \mathcal{U};\ y_{ij}, (i,j) \in \mathcal{R}) = x \tag{A.172}$$

and f is the length of the shortest path and may be calculated by the Dijkstra algorithm (Dijkstra [24]) when the weights are y_{ij} if $(i,j) \in \mathcal{R}$ and $\Upsilon_{ij}^{-1}(\alpha)$ if $(i,j) \in \mathcal{U}$, respectively.

Remark A.18: If the uncertain random network becomes a random network, then the shortest path distribution is

$$\Phi(x) = \int_{f(y_{ij},(i,j)\in\mathcal{R})\le x} \prod_{(i,j)\in\mathcal{R}} \mathrm{d}\Psi_{ij}(y_{ij}). \tag{A.173}$$

Remark A.19: (Gao [44]) If the uncertain random network becomes an uncertain network, then the inverse shortest path distribution is

$$\Phi^{-1}(\alpha) = f(\Upsilon_{ij}^{-1}(\alpha), (i,j) \in \mathcal{U}). \tag{A.174}$$

Exercise A.27: (Sheng-Gao [137]) Maximum flow problem is to find a flow with maximum value from a source node to a destination node in an uncertain random network. What is the maximum flow distribution?

A.13 Uncertain Random Process

Uncertain random process is a sequence of uncertain random variables indexed by time. A formal definition is given below.

Definition A.13 *(Gao-Yao [30]) Let* $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ *be a chance space and let* T *be a totally ordered set (e.g. time). An uncertain random process is a function* $X_t(\gamma,\omega)$ *from* $T\times(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ *to the set of real numbers such that* $\{X_t \in B\}$ *is an event in* $\mathcal{L}\times\mathcal{A}$ *for any Borel set* B *of real numbers at each time* t.

Example A.7: A stochastic process is a sequence of random variables indexed by time, and then is a special type of uncertain random process.

Example A.8: An uncertain process is a sequence of uncertain variables indexed by time, and then is a special type of uncertain random process.

Example A.9: Let Y_t be a stochastic process, and let Z_t be an uncertain process. If f is a measurable function, then

$$X_t = f(Y_t, Z_t) \tag{A.175}$$

is an uncertain random process.

Definition A.14 *(Gao-Yao [30]) Let* $\eta_1,\eta_2,\cdots$ *be iid random variables, let* $\tau_1,\tau_2,\cdots$ *be iid uncertain variables, and let* f *be a positive and strictly monotone function. Define* $S_0 = 0$ *and*

$$S_n = f(\eta_1,\tau_1) + f(\eta_2,\tau_2) + \cdots + f(\eta_n,\tau_n) \tag{A.176}$$

for $n \ge 1$. Then

$$N_t = \max_{n \ge 0} \left\{ n \;\middle|\; S_n \le t \right\} \tag{A.177}$$

is called an uncertain random renewal process with interarrival times $f(\eta_1, \tau_1)$, $f(\eta_2, \tau_2), \cdots$

Theorem A.36 *(Gao-Yao [30]) Let $\eta_1, \eta_2, \cdots$ be iid random variables with a common probability distribution Ψ, let $\tau_1, \tau_2, \cdots$ be iid uncertain variables, and let f be a positive and strictly monotone function. Assume N_t is an uncertain random renewal process with interarrival times $f(\eta_1, \tau_1), f(\eta_2, \tau_2), \cdots$ Then the average renewal number*

$$\frac{N_t}{t} \to \left(\int_{-\infty}^{+\infty} f(y, \tau_1) \mathrm{d}\Psi(y) \right)^{-1} \tag{A.178}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: Write $S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n)$ for all $n \ge 1$. Let x be a continuous point of the uncertainty distribution of

$$\left(\int_{-\infty}^{+\infty} f(y, \tau_1) \mathrm{d}\Psi(y) \right)^{-1}.$$

It is clear that $1/x$ is a continuous point of the uncertainty distribution of

$$\int_{-\infty}^{+\infty} f(y, \tau_1) \mathrm{d}\Psi(y).$$

At first, it follows from the definition of uncertain random renewal process that

$$\mathrm{Ch}\left\{ \frac{N_t}{t} \le x \right\} = \mathrm{Ch}\left\{ S_{\lfloor tx \rfloor + 1} > t \right\} = \mathrm{Ch}\left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} > \frac{t}{\lfloor tx \rfloor + 1} \right\}$$

where $\lfloor tx \rfloor$ represents the maximal integer less than or equal to tx. Since $\lfloor tx \rfloor \le tx < \lfloor tx \rfloor + 1$, we immediately have

$$\frac{\lfloor tx \rfloor}{\lfloor tx \rfloor + 1} \cdot \frac{1}{x} \le \frac{t}{\lfloor tx \rfloor + 1} < \frac{1}{x}$$

and then

$$\mathrm{Ch}\left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} > \frac{1}{x} \right\} \le \mathrm{Ch}\left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor + 1} > \frac{t}{\lfloor tx \rfloor + 1} \right\} \le \mathrm{Ch}\left\{ \frac{S_{\lfloor tx \rfloor + 1}}{\lfloor tx \rfloor} > \frac{1}{x} \right\}.$$

It follows from the law of large numbers for uncertain random variables that

$$\begin{aligned}\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}>\frac{1}{x}\right\}&=1-\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}\le\frac{1}{x}\right\}\\&=1-\mathcal{M}\left\{\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\le\frac{1}{x}\right\}\\&=\mathcal{M}\left\{\left(\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1}\le x\right\}\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor}>\frac{1}{x}\right\}&=1-\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{\lfloor tx\rfloor+1}{\lfloor tx\rfloor}\cdot\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}\le\frac{1}{x}\right\}\\&=1-\mathcal{M}\left\{\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\le\frac{1}{x}\right\}\\&=\mathcal{M}\left\{\left(\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1}\le x\right\}.\end{aligned}$$

From the above three relations we get

$$\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}>\frac{t}{\lfloor tx\rfloor+1}\right\}=\mathcal{M}\left\{\left(\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1}\le x\right\}$$

and then

$$\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{N_t}{t}\le x\right\}=\mathcal{M}\left\{\left(\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1}\le x\right\}.$$

The theorem is thus verified.

Exercise A.28: Let $\eta_1,\eta_2,\cdots$ be iid positive random variables, and let $\tau_1,\tau_2,\cdots$ be iid positive uncertain variables. Assume N_t is an uncertain random renewal process with interarrival times $\eta_1+\tau_1,\eta_2+\tau_2,\cdots$ Show that

$$\frac{N_t}{t}\to\frac{1}{E[\eta_1]+\tau_1}\tag{A.179}$$

in the sense of convergence in distribution as $t\to\infty$.

Exercise A.29: Let $\eta_1,\eta_2,\cdots$ be iid positive random variables, and let $\tau_1,\tau_2,\cdots$ be iid positive uncertain variables. Assume N_t is an uncertain random renewal process with interarrival times $\eta_1\tau_1,\eta_2\tau_2,\cdots$ Show that

$$\frac{N_t}{t}\to\frac{1}{E[\eta_1]\tau_1}\tag{A.180}$$

in the sense of convergence in distribution as $t\to\infty$.

Theorem A.37 *(Yao-Zhou [180]) Let $\eta_1, \eta_2, \cdots$ be iid random interarrival times, and let $\tau_1, \tau_2, \cdots$ be iid uncertain rewards. Assume N_t is a stochastic renewal process with interarrival times $\eta_1, \eta_2, \cdots$ Then*

$$R_t = \sum_{i=1}^{N_t} \tau_i \tag{A.181}$$

is an uncertain random renewal reward process, and

$$\frac{R_t}{t} \to \frac{\tau_1}{E[\eta_1]} \tag{A.182}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: Let Υ denote the uncertainty distribution of τ_1. Then for each realization of N_t, the uncertain variable

$$\frac{1}{N_t} \sum_{i=1}^{N_t} \tau_i$$

follows the uncertainty distribution Υ. In addition, by the definition of chance distribution, we have

$$\begin{aligned}
\mathrm{Ch}\left\{\frac{R_t}{t} \le x\right\} &= \int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{R_t}{t} \le x\right\} \ge r\right\} \mathrm{d}r \\
&= \int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{1}{N_t} \sum_{i=1}^{N_t} \tau_i \le \frac{tx}{N_t}\right\} \ge r\right\} \mathrm{d}r \\
&= \int_0^1 \Pr\left\{\Upsilon\left(\frac{tx}{N_t}\right) \ge r\right\} \mathrm{d}r
\end{aligned}$$

for any real number x. Since N_t is a stochastic renewal process with iid interarrival times $\eta_1, \eta_2, \cdots$, we have

$$\frac{t}{N_t} \to E[\eta_1], \quad a.s.$$

as $t \to \infty$. It follows from the Lebesgue domain convergence theorem that

$$\begin{aligned}
\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{R_t}{t} \le x\right\} &= \lim_{t\to\infty} \int_0^1 \Pr\left\{\Upsilon\left(\frac{tx}{N_t}\right) \ge r\right\} \mathrm{d}r \\
&= \int_0^1 \Pr\left\{\Upsilon(E[\eta_1]x) \ge r\right\} \mathrm{d}r = \Upsilon(E[\eta_1]x)
\end{aligned}$$

that is just the uncertainty distribution of $\tau_1/E[\eta_1]$. The theorem is thus proved.

Theorem A.38 *(Yao-Zhou [185]) Let $\eta_1, \eta_2, \cdots$ be iid random rewards, and let $\tau_1, \tau_2, \cdots$ be iid uncertain interarrival times. Assume N_t is an uncertain renewal process with interarrival times $\tau_1, \tau_2, \cdots$ Then*

$$R_t = \sum_{i=1}^{N_t} \eta_i \tag{A.183}$$

is an uncertain random renewal reward process, and

$$\frac{R_t}{t} \to \frac{E[\eta_1]}{\tau_1} \tag{A.184}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: Let Υ denote the uncertainty distribution of τ_1. It follows from the definition of chance distribution that for any real number x, we have

$$\begin{aligned} \mathrm{Ch}\left\{\frac{R_t}{t} \le x\right\} &= \int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{R_t}{t} \le x\right\} \ge r\right\} \mathrm{d}r \\ &= \int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{1}{x} \cdot \frac{1}{N_t}\sum_{i=1}^{N_t} \eta_i \le \frac{t}{N_t}\right\} \ge r\right\} \mathrm{d}r. \end{aligned}$$

Since N_t is an uncertain renewal process with iid interarrival times $\tau_1, \tau_2, \cdots$, by using Theorem 12.3, we have

$$\frac{t}{N_t} \to \tau_1$$

in the sense of convergence in distribution as $t \to \infty$. In addition, for each realization of N_t, the law of large numbers for random variables says

$$\frac{1}{N_t}\sum_{i=1}^{N_t} \eta_i \to E[\eta_1], \quad a.s.$$

as $t \to \infty$ for each number x. It follows from the Lebesgue domain convergence theorem that

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{R_t}{t} \le x\right\} = \int_0^1 \Pr\left\{1 - \Upsilon\left(\frac{E[\eta_1]}{x}\right) \ge r\right\} \mathrm{d}r = 1 - \Upsilon\left(\frac{E[\eta_1]}{x}\right)$$

that is just the uncertainty distribution of $E[\eta_1]/\tau_1$. The theorem is thus proved.

Theorem A.39 *(Yao-Gao [176]) Let $\eta_1, \eta_2, \cdots$ be iid random on-times, and let $\tau_1, \tau_2, \cdots$ be iid uncertain off-times. Assume N_t is an uncertain random* *renewal process with interarrival times* $\eta_1+\tau_1, \eta_2+\tau_2, \cdots$ *Then*

$$A_t = \begin{cases} t - \displaystyle\sum_{i=1}^{N_t} \tau_i, & \text{if } \displaystyle\sum_{i=1}^{N_t}(\eta_i+\tau_i) \le t < \sum_{i=1}^{N_t}(\eta_i+\tau_i) + \eta_{N_t+1} \\ \displaystyle\sum_{i=1}^{N_t+1} \eta_i, & \text{if } \displaystyle\sum_{i=1}^{N_t}(\eta_i+\tau_i) + \eta_{N_t+1} \le t < \sum_{i=1}^{N_t+1}(\eta_i+\tau_i) \end{cases} \tag{A.185}$$

is an uncertain random alternating renewal process (i.e., the total time at which the system is on up to time t*), and*

$$\frac{A_t}{t} \to \frac{E[\eta_1]}{E[\eta_1]+\tau_1} \tag{A.186}$$

in the sense of convergence in distribution as $t \to \infty$*.*

Proof: Let Φ denote the uncertainty distribution of τ_1, and let Υ be the uncertainty distribution of $E[\eta_1]/(E[\eta_1]+\tau_1)$. Then at each continuity point x of Υ, we have

$$\begin{aligned} \Upsilon(x) &= \mathcal{M}\left\{\frac{E[\eta_1]}{E[\eta_1]+\tau_1} \le x\right\} = \mathcal{M}\left\{\tau_1 \ge \frac{E[\eta_1](1-x)}{x}\right\} \\ &= 1 - \mathcal{M}\left\{\tau_1 < \frac{E[\eta_1](1-x)}{x}\right\} = 1 - \Phi\left(\frac{E[\eta_1](1-x)}{x}\right). \end{aligned}$$

On the one hand, by the Lebesgue dominated convergence theorem and the continuity of probability measure, we have

$$\begin{aligned} \lim_{t\to\infty} \mathrm{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\eta_i \le x\right\} &= \lim_{t\to\infty}\int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\eta_i \le x\right\} \ge r\right\} \mathrm{d}r \\ &= \int_0^1 \lim_{t\to\infty} \Pr\left\{\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\eta_i \le x\right\} \ge r\right\} \mathrm{d}r \\ &= \int_0^1 \Pr\left\{\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\eta_i \le x\right\} \ge r\right\} \mathrm{d}r. \end{aligned}$$

Note that

$$
\begin{aligned}
\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\eta_i \le x\right\} &= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\frac{1}{t}\sum_{i=1}^{k}\eta_i \le x\right)\cap (N_t = k)\right\}\\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}\eta_i \le tx\right)\cap \left(\sum_{i=1}^{k+1}(\eta_i+\tau_i) > t\right)\right\}\\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}\eta_i \le tx\right)\cap \left(tx+\eta_{k+1}+\sum_{i=1}^{k+1}\tau_i > t\right)\right\}\\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}(k \le N_{tx}^*)\cap \left(\frac{\eta_{k+1}}{t}+\frac{1}{t}\sum_{i=1}^{k+1}\tau_i > 1-x\right)\right\}
\end{aligned}
$$

where N_t^* is a stochastic renewal process with random interarrival times $\eta_1, \eta_2, \cdots$ Since

$$\frac{\eta_{k+1}}{t} \to 0 \text{ as } t \to \infty$$

and

$$\sum_{i=1}^{k+1}\tau_i \sim (k+1)\tau_1,$$

we have

$$
\begin{aligned}
\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\eta_i \le x\right\} &\le \lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=0}^{\infty}(k \le N_{tx}^*)\cap\left(\tau_1 > \frac{t-tx}{k+1}\right)\right\}\\
&= \lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=0}^{N_{tx}^*}\left(\tau_1 > \frac{t-tx}{k+1}\right)\right\}\\
&= \lim_{t\to\infty}\mathcal{M}\left\{\tau_1 > \frac{t-tx}{N_{tx}^*+1}\right\}\\
&= 1-\lim_{t\to\infty}\Phi\left(\frac{t-tx}{N_{tx}^*+1}\right).
\end{aligned}
$$

By the elementary renewal theorem in probability, we have

$$\frac{N_{tx}^*}{tx} \to \frac{1}{E[\eta_1]}, \quad \text{a.s.}$$

as $t \to \infty$, and then

$$\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\eta_i \le x\right\} \le 1-\Phi\left(\frac{E[\eta_1](1-x)}{x}\right) = \Upsilon(x).$$

Thus

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\eta_i \le x\right\} \le \int_0^1 \Pr\left\{\Upsilon(x)\ge r\right\}\mathrm{d}r = \Upsilon(x). \tag{A.187}$$

On the other hand, by the Lebesgue dominated convergence theorem and the continuity of probability measure, we have

$$\begin{aligned}
\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i > x\right\} &= \lim_{t\to\infty}\int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i > x\right\}\ge r\right\}\mathrm{d}r\\
&= \int_0^1 \lim_{t\to\infty}\Pr\left\{\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i > x\right\}\ge r\right\}\mathrm{d}r\\
&= \int_0^1 \Pr\left\{\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i > x\right\}\ge r\right\}\mathrm{d}r.
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i > x\right\}\\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\frac{1}{t}\sum_{i=1}^{k+1}\eta_i > x\right)\cap(N_t = k)\right\}\\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k+1}\eta_i > tx\right)\cap\left(\sum_{i=1}^{k}(\eta_i+\tau_i)\le t\right)\right\}\\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k+1}\eta_i > tx\right)\cap\left(tx-\eta_{k+1}+\sum_{i=1}^{k}\tau_i\le t\right)\right\}\\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}(N_{tx}^*\le k)\cap\left(\frac{1}{t}\sum_{i=1}^{k}\tau_i-\frac{\eta_{k+1}}{t}\le 1-x\right)\right\}.
\end{aligned}$$

Since

$$\sum_{i=1}^{k}\tau_i \sim k\tau_1$$

and

$$\frac{\eta_{k+1}}{t}\to 0 \text{ as } t\to\infty,$$

we have

$$\begin{aligned}\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i > x\right\} &\le \lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=0}^{\infty}(N^*_{tx}\le k)\cap\left(\frac{1}{t}\sum_{i=1}^{k}\tau_i\le 1-x\right)\right\}\\ &= \lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=N^*_{tx}}^{\infty}\left(\tau_1\le\frac{t-tx}{k}\right)\right\}\\ &= \lim_{t\to\infty}\mathcal{M}\left\{\tau_1\le\frac{t-tx}{N^*_{tx}}\right\}\\ &= \lim_{t\to\infty}\Phi\left(\frac{t-tx}{N^*_{tx}}\right).\end{aligned}$$

By the elementary renewal theorem, we have

$$\frac{N^*_{tx}}{tx}\to\frac{1}{E[\eta_1]},\quad \text{a.s.}$$

as $t\to\infty$, and then

$$\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i > x\right\}\le\Phi\left(\frac{E[\eta_1](1-x)}{x}\right)=1-\Upsilon(x).$$

Thus

$$\lim_{t\to\infty}\text{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i > x\right\}\le\int_0^1\Pr\left\{1-\Upsilon(x)\ge r\right\}\mathrm{d}r=1-\Upsilon(x).$$

By using the duality property of chance measure, we get

$$\lim_{t\to\infty}\text{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i\le x\right\}\ge\Upsilon(x). \tag{A.188}$$

Since

$$\frac{1}{t}\sum_{i=1}^{N_t}\eta_i\le\frac{A_t}{t}\le\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i,$$

we obtain

$$\text{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\eta_i\le x\right\}\le\text{Ch}\left\{\frac{A_t}{t}\le x\right\}\le\text{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\eta_i\le x\right\}.$$

It follows from (A.187) and (A.188) that

$$\lim_{t\to\infty}\text{Ch}\left\{\frac{A_t}{t}\le x\right\}=\Upsilon(x).$$

Hence the availability rate A_t/t converges in distribution to $E[\eta_1]/(E[\eta_1]+\tau_1)$ as $t\to\infty$. The theorem is proved.

Theorem A.40 *(Yao-Gao [176]) Let $\tau_1, \tau_2, \cdots$ be iid uncertain on-times, and let $\eta_1, \eta_2, \cdots$ be iid random off-times. Assume N_t is an uncertain random renewal process with interarrival times $\tau_1 + \eta_1, \tau_2 + \eta_2, \cdots$ Then*

$$A_t = \begin{cases} t - \sum_{i=1}^{N_t} \eta_i, & \textit{if } \sum_{i=1}^{N_t}(\tau_i + \eta_i) \le t < \sum_{i=1}^{N_t}(\tau_i + \eta_i) + \tau_{N_t+1} \\ \sum_{i=1}^{N_t+1} \tau_i, & \textit{if } \sum_{i=1}^{N_t}(\tau_i + \eta_i) + \tau_{N_t+1} \le t < \sum_{i=1}^{N_t+1}(\tau_i + \eta_i) \end{cases} \tag{A.189}$$

is an uncertain random alternating renewal process (i.e., the total time at which the system is on up to time t), and

$$\frac{A_t}{t} \to \frac{\tau_1}{\tau_1 + E[\eta_1]} \tag{A.190}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: Let Φ denote the uncertainty distribution of τ_1, and let Υ be the uncertainty distribution of $\tau_1/(\tau_1 + E[\eta_1])$. Then at each continuity point x of Υ, we have

$$\Upsilon(x) = \mathcal{M}\left\{\frac{\tau_1}{\tau_1 + E[\eta_1]} \le x\right\} = \mathcal{M}\left\{\tau_1 \le \frac{E[\eta_1]x}{1-x}\right\} = \Phi\left(\frac{E[\eta_1]x}{1-x}\right).$$

On the one hand, by the Lebesgue dominated convergence theorem and the continuity of probability measure, we have

$$\begin{aligned} \lim_{t\to\infty} \text{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t} \tau_i \le x\right\} &= \lim_{t\to\infty} \int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t} \tau_i \le x\right\} \ge r\right\} \mathrm{d}r \\ &= \int_0^1 \lim_{t\to\infty} \Pr\left\{\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t} \tau_i \le x\right\} \ge r\right\} \mathrm{d}r \\ &= \int_0^1 \Pr\left\{\lim_{t\to\infty} \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t} \tau_i \le x\right\} \ge r\right\} \mathrm{d}r. \end{aligned}$$

Note that

$$\begin{aligned}
&\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\tau_i \le x\right\}\\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\frac{1}{t}\sum_{i=1}^{k}\tau_i \le x\right)\cap (N_t = k)\right\}\\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}\tau_i \le tx\right)\cap \left(\sum_{i=1}^{k+1}(\tau_i+\eta_i) > t\right)\right\}\\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}\tau_i \le tx\right)\cap \left(tx+\tau_{k+1}+\sum_{i=1}^{k+1}\eta_i > t\right)\right\}\\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}\tau_i \le tx\right)\cap \left(\frac{\tau_{k+1}}{t}+\frac{1}{t}\sum_{i=1}^{k+1}\eta_i > 1-x\right)\right\}.
\end{aligned}$$

Since

$$\sum_{i=1}^{k}\tau_i \sim k\tau_1$$

and

$$\frac{\tau_{k+1}}{t} \to 0 \text{ as } t\to\infty,$$

we have

$$\begin{aligned}
&\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\tau_i \le x\right\}\\
&\le \lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\tau_1 \le \frac{tx}{k}\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k+1}\eta_i > 1-x\right)\right\}\\
&= \lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\tau_1 \le \frac{tx}{k}\right)\cap\left(N^*_{t-tx} \le k\right)\right\}\\
&= \lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=N^*_{t-tx}}^{\infty}\left(\tau_1 \le \frac{tx}{k}\right)\right\}\\
&= \lim_{t\to\infty}\mathcal{M}\left\{\tau_1 \le \frac{tx}{N^*_{t-tx}}\right\}\\
&= \lim_{t\to\infty}\Phi\left(\frac{tx}{N^*_{t-tx}}\right)
\end{aligned}$$

where N_t^* is a stochastic renewal process with random interarrival times $\eta_1, \eta_2, \cdots$ By the elementary renewal theorem, we have

$$\frac{N^*_{t-tx}}{t-tx} \to \frac{1}{E[\eta_1]}, \quad \text{a.s.}$$

as $t \to \infty$, and then

$$\lim_{t\to\infty} \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t} \tau_i \le x\right\} \le \Phi\left(\frac{E[\eta_1]x}{1-x}\right) = \Upsilon(x).$$

Thus

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t} \tau_i \le x\right\} \le \int_0^1 \Pr\left\{\Upsilon(x) \ge r\right\} \mathrm{d}r = \Upsilon(x). \tag{A.191}$$

On the other hand, by the Lebesgue dominated convergence theorem and the continuity of probability measure, we have

$$\begin{aligned}
\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1} \tau_i > x\right\} &= \lim_{t\to\infty}\int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1} \tau_i > x\right\} \ge r\right\} \mathrm{d}r \\
&= \int_0^1 \lim_{t\to\infty} \Pr\left\{\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1} \tau_i > x\right\} \ge r\right\} \mathrm{d}r \\
&= \int_0^1 \Pr\left\{\lim_{t\to\infty} \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1} \tau_i > x\right\} \ge r\right\} \mathrm{d}r.
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1} \tau_i > x\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\frac{1}{t}\sum_{i=1}^{k+1}\tau_i > x\right) \cap (N_t = k)\right\} \\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k+1}\tau_i > tx\right) \cap \left(\sum_{i=1}^{k}(\tau_i + \eta_i) \le t\right)\right\} \\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k+1}\tau_i > tx\right) \cap \left(tx - \tau_{k+1} + \sum_{i=1}^{k}\eta_i \le t\right)\right\} \\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k+1}\tau_i > tx\right) \cap \left(\frac{1}{t}\sum_{i=1}^{k}\eta_i - \frac{\tau_{k+1}}{t} \le 1 - x\right)\right\}.
\end{aligned}$$

Since

$$\sum_{i=1}^{k+1} \tau_i \sim (k+1)\tau_1$$

and

$$\frac{\tau_{k+1}}{t} \to 0 \text{ as } t \to \infty,$$

we have

$$\begin{aligned}
&\lim_{t\to\infty} \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1} \tau_i > x\right\} \\
&\le \lim_{t\to\infty} \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\tau_1 > \frac{tx}{k+1}\right) \cap \left(\frac{1}{t}\sum_{i=1}^{k} \tau_i \le 1-x\right)\right\} \\
&= \lim_{t\to\infty} \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\tau_1 > \frac{tx}{k+1}\right) \cap \left(N^*_{t-tx} \ge k\right)\right\} \\
&= \lim_{t\to\infty} \mathcal{M}\left\{\bigcup_{k=0}^{N^*_{t-tx}}\left(\tau_1 > \frac{tx}{k+1}\right)\right\} \\
&= \lim_{t\to\infty} \mathcal{M}\left\{\tau_1 > \frac{tx}{N^*_{t-tx}+1}\right\} \\
&= 1 - \lim_{t\to\infty} \Phi\left(\frac{tx}{N^*_{tx}+1}\right).
\end{aligned}$$

By the elementary renewal theorem, we have

$$\frac{N^*_{t-tx}}{t-tx} \to \frac{1}{E[\eta_1]}, \quad \text{a.s.}$$

as $t \to \infty$, and then

$$\lim_{t\to\infty} \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1} \tau_i > x\right\} \le 1 - \Phi\left(\frac{E[\eta_1]x}{1-x}\right) = 1 - \Upsilon(x).$$

Thus

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1} \tau_i > x\right\} \le \int_0^1 \Pr\{1-\Upsilon(x) \ge r\}\,\mathrm{d}r = 1 - \Upsilon(x).$$

By using the duality property of chance measure, we get

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1} \tau_i \le x\right\} \ge \Upsilon(x). \tag{A.192}$$

Since

$$\frac{1}{t}\sum_{i=1}^{N_t}\tau_i \le \frac{A_t}{t} \le \frac{1}{t}\sum_{i=1}^{N_t+1}\tau_i,$$

we obtain

$$\text{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\tau_i \le x\right\} \le \text{Ch}\left\{\frac{A_t}{t} \le x\right\} \le \text{Ch}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\tau_i \le x\right\}.$$

It follows from (A.191) and (A.192) that

$$\lim_{t\to\infty}\text{Ch}\left\{\frac{A_t}{t} \le x\right\} = \Upsilon(x).$$

Hence the availability rate A_t/t converges in distribution to $\tau/(\tau_1 + E[\eta_1])$ as $t \to \infty$. The theorem is proved.

A.14 Bibliographic Notes

Probability theory was developed by Kolmogorov [69] in 1933 for modelling frequencies, while uncertainty theory was founded by Liu [76] in 2007 for modelling belief degrees. However, in many cases, uncertainty and randomness simultaneously appear in a complex system. In order to describe this phenomenon, uncertain random variable was initialized by Liu [105] in 2013 with the concepts of chance measure and chance distribution. As an important contribution, Liu [106] presented an operational law of uncertain random variables. Furthermore, Yao-Gao [179], Gao-Sheng [32] and Gao-Ralescu [39] verified some laws of large numbers for uncertain random variables.

Stochastic programming was first studied by Dantzig [21] in 1965, while uncertain programming was first proposed by Liu [78] in 2009. In order to model optimization problems with not only uncertainty but also randomness, uncertain random programming was founded by Liu [106] in 2013. As extensions, Zhou-Yang-Wang [202] proposed uncertain random multiobjective programming for optimizing multiple, noncommensurable and conflicting objectives, Qin [125] proposed uncertain random goal programming in order to satisfy as many goals as possible in the order specified, and Ke-Su-Ni [65] proposed uncertain random multilevel programming for studying decentralized decision systems in which the leader and followers may have their own decision variables and objective functions. After that, uncertain random programming was developed steadily and applied widely.

Probabilistic risk analysis was dated back to 1952 when Roy [129] proposed his safety-first criterion for portfolio selection. Another important contribution is the probabilistic value-at-risk methodology developed by Morgan [114] in 1996. On the other hand, uncertain risk analysis was proposed by Liu [82] in 2010 for evaluating the risk index that is the uncertain measure of an uncertain system being loss-positive. More generally, in order to quantify the risk of uncertain random systems, Liu-Ralescu [107] invented the tool of uncertain random risk analysis in 2014. Furthermore, the value-at-risk methodology was presented by Liu-Ralescu [109], and the expected loss methodology was investigated by Liu-Ralescu [111] for dealing with uncertain random systems.

Probabilistic reliability analysis was traced back to 1944 when Pugsley [123] first proposed structural accident rates for the aeronautics industry. Nowadays, probabilistic reliability analysis has become a widely used discipline. As a new methodology, uncertain reliability analysis was developed by Liu [82] in 2010 for evaluating the reliability index. More generally, for dealing with uncertain random systems, Wen-Kang [153] presented the tool of uncertain random reliability analysis and defined the reliability index in 2016. In addition, Gao-Yao [34] analyzed the importance index in uncertain random system.

Random graph was defined by Erdős-Rényi [28] in 1959 and independently by Gilbert [50] at nearly the same time. As an alternative, uncertain graph was proposed by Gao-Gao [42] in 2013 via uncertainty theory. Assuming some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure, Liu [92] defined the concept of uncertain random graph and analyzed the connectivity index in 2014. After that, Zhang-Peng-Li [194] discussed the Euler index of uncertain random graph.

Random network was first investigated by Frank-Hakimi [29] in 1965 for modelling communication network with random capacities. From then on, the random network was well developed and widely applied. As a breakthrough approach, uncertain network was first explored by Liu [83] in 2010 for modelling project scheduling problem with uncertain duration times. More generally, assuming some weights are random variables and others are uncertain variables, Liu [92] initialized the concept of uncertain random network and discussed the shortest path problem in 2014. Following that, uncertain random network was explored by many researchers. For example, Sheng-Gao [137] investigated the maximum flow problem, and Sheng-Qin-Shi [140] dealt with the minimum spanning tree problem of uncertain random network.

One of the earliest investigations of stochastic process was Bachelier [1] in 1900, and the study of uncertain process was started by Liu [77] in 2008. In order to deal with uncertain random phenomenon evolving in time, Gao-Yao [30] presented an uncertain random process in the light of chance theory in 2015. Gao-Yao [30] also proposed an uncertain random renewal process. As extensions, Yao-Zhou [180] discussed an uncertain random renewal reward process, and Yao-Gao [176] investigated an uncertain random alternating renewal process.

Appendix B

Frequently Asked Questions

This appendix will answer some frequently asked questions related to probability theory and uncertainty theory as well as their applications. This appendix will also show why fuzzy set is a wrong model in both theory and practice. Finally, I will clarify what uncertainty is.

B.1 What is the meaning that an object follows the laws of probability theory?

We say an object (e.g. frequency) follows the laws of probability theory if it meets not only the three axioms (Kolmogorov [69]) but also the product probability theorem of probability theory:

Axiom 1 *(Normality Axiom)* $\Pr\{\Omega\} = 1$ *for the universal set* Ω;

Axiom 2 *(Nonnegativity Axiom)* $\Pr\{A\} \geq 0$ *for any event* A;

Axiom 3 *(Additivity Axiom) For every countable sequence of mutually disjoint events* $A_1, A_2, \cdots$, *we have*

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \Pr\{A_i\}; \tag{B.1}$$

Theorem *(Product Probability) Let* $(\Omega_k, \mathcal{A}_k, \Pr_k)$ *be probability spaces for* $k = 1, 2, \cdots$ *Then there is a unique probability measure* $\Pr$ *such that*

$$\Pr\left\{\prod_{k=1}^{\infty} A_k\right\} = \prod_{k=1}^{\infty} \Pr_k\{A_k\} \tag{B.2}$$

where A_k *are arbitrarily chosen events from* $\mathcal{A}_k$ *for* $k = 1, 2, \cdots$, *respectively.*

It is easy for us to understand why we need to justify that the object meets the three axioms. However, some readers may wonder why we also need to justify that the object meets the product probability theorem. The reason is that *product probability theorem cannot be deduced from Kolmogorov's axioms* except we presuppose that the product probability meets the three axioms. In other words, an object does not necessarily satisfy the product probability theorem if it is only justified to meet the three axioms. Would that surprise you?

Please keep in mind that "an object follows the laws of probability theory" is equivalent to "an object meets the three axioms plus the product probability theorem". This assertion is stronger than "an object meets the three axioms of Kolmogorov". In other words, the three axioms do not ensure that an object follows the laws of probability theory.

There exist two broad categories of interpretations of probability, one is *frequency interpretation* and the other is *belief interpretation*. The frequency interpretation takes the probability to be the frequency with which an event happens (Venn [144], Reichenbach [127], von Mises [145]), while the belief interpretation takes the probability to be the degree to which we believe an event will happen (Ramsey [126], de Finetti [22], Savage [131]).

The debate between the two interpretations has been lasting from the nineteenth century. Personally, I agree with the frequency interpretation, but strongly oppose the belief interpretation of probability because frequency follows the laws of probability theory but belief degree does not. The detailed reasons will be given in the following a few sections.

B.2 Why does frequency follow the laws of probability theory?

In order to show that the frequency follows the laws of probability theory, we must verify that the frequency meets not only the three axioms of Kolmogorov but also the product probability theorem.

First, the frequency of the universal set takes value 1 because the universal set always happens. Thus the frequency meets the normality axiom. Second, it is obvious that the frequency is a number between 0 and 1. Thus the frequency of any event is nonnegative, and the frequency meets the nonnegativity axiom. Third, for any disjoint events A and B, if A happens α times and B happens β times (in percentage), it is clear that the union $A\cup B$ happens $\alpha+\beta$ times. This means the frequency is additive and then meets the additivity axiom. Finally, numerous experiments showed that if A and B are two events from different probability spaces (essentially they come from two different experiments) and happen α and β times, respectively, then the product $A\times B$ happens $\alpha\times\beta$ times. See Figure B.1. Thus the frequency meets the product probability theorem. Hence the frequency does follow the laws of probability theory. In fact, frequency is the only empirical basis for probability theory.

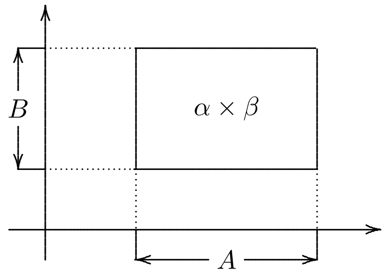


Figure B.1: Let A and B be two events from different probability spaces (essentially they come from two different experiments). If A happens α times and B happens β times, then the product $A \times B$ happens $\alpha \times \beta$ times, where α and β are understood as percentage numbers.

B.3 Why is probability theory not suitable for modelling belief degree?

In order to obtain the belief degree of some event, the decision maker needs to launch a consultation process with a domain expert. The decision maker is the user of belief degree while the domain expert is the holder. For justifying whether probability theory is suitable for modelling belief degree or not, we must check if the belief degree follows the laws of probability theory.

First, "1" means "complete belief" and we cannot be in more belief than "complete belief". This means the belief degree of any event cannot exceed 1. Furthermore, the belief degree of the universal set takes value 1 because it is completely believable. Hence the belief degree meets the normality axiom of probability theory.

Second, the belief degree meets the nonnegativity axiom because "0" means "complete disbelief" and we cannot be in more disbelief than "complete disbelief".

Third, de Finetti [22] interpreted the belief degree of an event as the fair betting ratio (price/stake) of a bet that offers \$1 if the event happens and nothing otherwise. For example, if the domain expert thinks the belief degree of an event A is α, then the price of the bet about A is $\alpha \times 100$¢. Here the word "fair" means both the domain expert and the decision maker are willing to either buy or sell this bet at this price.

Besides, Ramsey [126] suggested a Dutch book argument[1] that says *the*

[1] A Dutch book in a betting market is a set of bets which guarantees a loss, regardless of the outcome of the gamble. For example, let A be a bet that offers \$1 if A happens, let B be a bet that offers \$1 if B happens, and let $A \vee B$ be a bet that offers \$1 if either A or B happens. If the prices of A, B and $A \vee B$ are 30¢, 40¢ and 80¢, respectively, and you (i)

belief degree is irrational if there exists a book that guarantees you a loss. For the moment, we are assumed to agree with it.

Let A_1 be a bet that offers \$1 if A_1 happens, and let A_2 be a bet that offers \$1 if A_2 happens. Assume the belief degrees of A_1 and A_2 are α_1 and α_2, respectively. This means the prices of A_1 and A_2 are $\$\alpha_1$ and $\$\alpha_2$, respectively. Now we consider the bet $A_1 \cup A_2$ that offers \$1 if either A_1 or A_2 happens, and write the belief degree of $A_1 \cup A_2$ by α. This means the price of $A_1 \cup A_2$ is $\$\alpha$. If $\alpha > \alpha_1 + \alpha_2$, then you (i) sell A_1, (ii) sell A_2, and (iii) buy $A_1 \cup A_2$. It is clear that you are guaranteed to lose $\alpha - \alpha_1 - \alpha_2 > 0$. Thus there exists a Dutch book and the assumption $\alpha > \alpha_1 + \alpha_2$ is irrational. If $\alpha < \alpha_1 + \alpha_2$, then you (i) buy A_1, (ii) buy A_2, and (iii) sell $A_1 \cup A_2$. It is clear that you are guaranteed to lose $\alpha_1 + \alpha_2 - \alpha > 0$. Thus there exists a Dutch book and the assumption $\alpha < \alpha_1 + \alpha_2$ is irrational. Hence we have to assume $\alpha = \alpha_1 + \alpha_2$ and the belief degree meets the additivity axiom (but this assertion is questionable because you cannot reverse "buy" and "sell" arbitrarily due to the unequal status of the decision maker and the domain expert).

Until now we have verified that the belief degree meets the three axioms of probability theory. Almost all subjectivists stop here and assert that belief degree follows the laws of probability theory. Unfortunately, the evidence is not enough for this conclusion because we have not verified whether belief degree meets the product probability theorem or not. In fact, it is impossible for us to prove belief degree meets the product probability theorem through the Dutch book argument.

Recall the example of truck-cross-over-bridge on Page 6. Let A_i represent that the ith bridge strengths are greater than 90 tons, $i = 1, 2, \cdots, 50$, respectively. For each i, since your belief degree for A_i is 75%, you are willing to pay 75¢ for the bet that offers \$1 if A_i happens. If the belief degree did follow the laws of probability theory, then it would be fair to pay

$$\underbrace{75\% \times 75\% \times \cdots \times 75\%}_{50} \times 100¢ \approx 0.00006¢ \tag{B.3}$$

for a bet that offers \$1 if $A_1 \times A_2 \times \cdots \times A_{50}$ happens. Notice that the odd is over a million and $A_1 \times A_2 \times \cdots \times A_{50}$ definitely happens because the real strengths of the 50 bridges range from 95 to 110 tons. All of us will be happy to bet on it. But who is willing to offer such a bet? It seems that no one does, and then the belief degree of $A_1 \times A_2 \times \cdots \times A_{50}$ is not the product of each individuals. Hence the belief degree does not follow the laws of probability theory.

It is thus concluded that the belief interpretation of probability is unacceptable. The main mistake of subjectivists is that they only justify the

sell A, (ii) sell B, and (iii) buy $A \vee B$, then you are guaranteed to lose 10¢ no matter what happens. Thus there exists a Dutch book, and the prices are considered to be irrational.

belief degree meets the three axioms of probability theory, but do not check if it meets the product probability theorem.

B.4 What goes wrong with Cox's theorem?

Some people affirm that *probability theory is the only legitimate approach.* Perhaps this misconception is rooted in Cox's theorem [18] that any measure of belief is "isomorphic" to a probability measure. However, uncertain measure is considered coherent but not isomorphic to any probability measure. What goes wrong with Cox's theorem? Personally I think that Cox's theorem presumes the truth value of conjunction $P \wedge Q$ is a twice differentiable function f of truth values of the two propositions P and Q, i.e.,

$$T(P \wedge Q) = f(T(P), T(Q)) \tag{B.4}$$

and then excludes uncertain measure from its start because the function $f(x, y) = x \wedge y$ used in uncertainty theory is not differentiable with respect to x and y. In fact, there does not exist any evidence that the truth value of conjunction is completely determined by the truth values of individual propositions, let alone a twice differentiable function.

On the one hand, it is recognized that probability theory is a legitimate approach to deal with the frequency. On the other hand, at any rate, it is impossible that probability theory is the unique one for modelling indeterminacy. In fact, it has been demonstrated in this book that uncertainty theory is successful to deal with belief degrees.

B.5 What is the difference between probability theory and uncertainty theory?

The difference between probability theory (Kolmogorov [69]) and uncertainty theory (Liu [76]) does not lie in whether the measures are additive or not, but how the product measures are defined. The product probability measure is the product of the probability measures of the individual events, i.e.,

$$\Pr\{\Lambda_1 \times \Lambda_2\} = \Pr\{\Lambda_1\} \times \Pr\{\Lambda_2\}, \tag{B.5}$$

while the product uncertain measure is the minimum of the uncertain measures of the individual events, i.e.,

$$\mathcal{M}\{\Lambda_1 \times \Lambda_2\} = \mathcal{M}\{\Lambda_1\} \wedge \mathcal{M}\{\Lambda_2\}. \tag{B.6}$$

Shortly, we may say that *probability theory is a "product" mathematical system, and uncertainty theory is a "minimum" mathematical system.* This difference implies that random variables and uncertain variables obey different operational laws.

Probability theory and uncertainty theory are complementary mathematical systems that provide two acceptable mathematical models to deal with the indeterminate world. Probability theory is a branch of mathematics for modelling frequencies, while uncertainty theory is a branch of mathematics for modelling belief degrees.

B.6 Why do I think fuzzy set theory is bad mathematics?

A fuzzy set is defined by its membership function μ which assigns to each element x a real number $\mu(x)$ in the interval $[0,1]$, where the value of $\mu(x)$ represents the grade of membership of x in the fuzzy set. This definition was given by Zadeh [189] in 1965. Since then, fuzzy set theory has been spread broadly. Although I strongly respect Professor Lotfi Zadeh's achievements, I have to declare that fuzzy set theory is bad mathematics.

A very strange phenomenon in academia is that different people have different fuzzy set theories. Even so, we have to admit that every version of fuzzy set theory contains at least the following four items. The first one is a fuzzy set ξ with membership function μ. The next one is a complement set ξ^c with membership function

$$\lambda(x) = 1 - \mu(x). \tag{B.7}$$

The third one is a possibility measure defined by the three axioms,

$$\text{Pos}\{\Omega\} = 1 \text{ for the universal set } \Omega, \tag{B.8}$$

$$\text{Pos}\{\emptyset\} = 0 \text{ for the empty set } \emptyset, \tag{B.9}$$

$$\text{Pos}\{\Lambda_1 \cup \Lambda_2\} = \text{Pos}\{\Lambda_1\} \vee \text{Pos}\{\Lambda_2\} \text{ for any events } \Lambda_1 \text{ and } \Lambda_2. \tag{B.10}$$

And the fourth one is a relation between membership function and possibility measure (Zadeh [190]),

$$\mu(x) = \text{Pos}\{x \in \xi\}. \tag{B.11}$$

Now for any point x, it is clear that $\{x \in \xi\}$ and $\{x \in \xi^c\}$ are opposite events[2], and then

$$\{x \in \xi\} \cup \{x \in \xi^c\} = \Omega. \tag{B.12}$$

On the one hand, by using the possibility axioms, we have

$$\text{Pos}\{x \in \xi\} \vee \text{Pos}\{x \in \xi^c\} = \text{Pos}\{\Omega\} = 1. \tag{B.13}$$

[2]Please do not challenge this proposition, otherwise the classical mathematics has to be completely rewritten. Perhaps some fuzzists insist that $\{x \in \xi\}$ and $\{x \in \xi^c\}$ are not opposite. Here I would like to advise them not to think so because it is in contradiction with that ξ^c has the membership function $\lambda(x) = 1 - \mu(x)$.

On the other hand, by using the relation (B.11), we have

$$\mathrm{Pos}\{x \in \xi\} = \mu(x), \tag{B.14}$$

$$\mathrm{Pos}\{x \in \xi^c\} = 1 - \mu(x). \tag{B.15}$$

It follows from (B.13), (B.14) and (B.15) that

$$\mu(x) \vee (1 - \mu(x)) = 1. \tag{B.16}$$

Hence

$$\mu(x) = 0 \text{ or } 1. \tag{B.17}$$

This result shows that the membership function μ can only be an indicator function of crisp set. In other words, only crisp sets can simultaneously satisfy (B.7)∼(B.11). In this sense, *fuzzy set theory collapses mathematically to classical set theory*. That is, fuzzy set theory is nothing but classical set theory.

Furthermore, it seems both in theory and practice that inclusion relation between fuzzy sets has to be needed. Thus fuzzy set theory also assumes a formula (Zadeh [190]),

$$\mathrm{Pos}\{\xi \subset B\} = \sup_{x \in B} \mu(x) \tag{B.18}$$

for any crisp set B. Now consider two crisp intervals $[1, 2]$ and $[2, 3]$. It is completely inacceptable in mathematical community that $[1, 2]$ is included in $[2, 3]$, i.e., the inclusion relation

$$[1, 2] \subset [2, 3] \tag{B.19}$$

is 100% wrong. Note that $[1, 2]$ is a special fuzzy set whose membership function is

$$\mu(x) = \begin{cases} 1, & \text{if } 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases} \tag{B.20}$$

It follows from the formula (B.18) that

$$\mathrm{Pos}\{[1, 2] \subset [2, 3]\} = \sup_{x \in [2,3]} \mu(x) = 1. \tag{B.21}$$

That is, fuzzy set theory says that $[1, 2] \subset [2, 3]$ is 100% right. Are you willing to accept this result? If not, then (B.18) is in conflict with the inclusion relation in classical set theory. In other words, nothing can simultaneously satisfy (B.7)∼(B.11) and (B.18). Therefore, *fuzzy set theory is not self-consistent in mathematics* and may lead to wrong results in practice.

Perhaps some fuzzists may argue that they never use possibility measure in fuzzy set theory. Here I would like to remind them that the membership degree $\mu(x)$ is just the possibility measure that the fuzzy set ξ contains the point x (i.e., x belongs to ξ). Please also keep in mind that we cannot distinguish fuzzy set from random set (Robbins [128] and Matheron [112]) and uncertain set (Liu [81]) if the underlying measures are not available.

From the above discussion, we can see that fuzzy set theory is not self-consistent in mathematics and may lead to wrong results in practice. Therefore, I would like to conclude that *fuzzy set theory is bad mathematics.* To express this more frankly, fuzzy set theory cannot be called mathematics. Can we improve fuzzy set theory? Yes, we can. But the change is so big that I have to give the revision a new name called uncertain set theory. See Chapter 8.

B.7 Why is fuzzy variable not suitable for modelling indeterminate quantity?

A fuzzy variable is a function from a possibility space to the set of real numbers (Nahmias [115]). Some people think that fuzzy variable is a suitable tool for modelling indeterminate quantity. Is it really true? Unfortunately, the answer is negative.

Let us reconsider the counterexample of truck-cross-over-bridge (Liu [85]). If the bridge strength is regarded as a fuzzy variable ξ, then we may assign it a membership function, say

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 80 \\ (x-80)/10, & \text{if } 80 \le x \le 90 \\ 1, & \text{if } 90 \le x \le 110 \\ (120-x)/10, & \text{if } 110 \le x \le 120 \\ 0, & \text{if } x \ge 120 \end{cases} \tag{B.22}$$

that is just the trapezoidal fuzzy variable $(80, 90, 110, 120)$. Please do not argue why I choose such a membership function because it is not important for the focus of debate. Based on the membership function μ and the definition of possibility measure

$$\text{Pos}\{\xi \in B\} = \sup_{x \in B} \mu(x), \tag{B.23}$$

it is easy for us to infer that

$$\text{Pos}\{\text{“bridge strength”} = 100\} = 1, \tag{B.24}$$

$$\text{Pos}\{\text{“bridge strength”} \ne 100\} = 1. \tag{B.25}$$

Thus we immediately conclude the following three propositions:

(a) the bridge strength is “exactly 100 tons” with possibility measure 1,

(b) the bridge strength is “not 100 tons” with possibility measure 1,

(c) “exactly 100 tons” is as possible as “not 100 tons”.

The first proposition says we are 100% sure that the bridge strength is "exactly 100 tons", neither less nor more. What a coincidence it should be! It is doubtless that the belief degree of "exactly 100 tons" is almost zero, and nobody is so naive to expect that "exactly 100 tons" is the true bridge strength. The second proposition sounds good. The third proposition says "exactly 100 tons" and "not 100 tons" have the same possibility measure. Thus we have to regard them "equally likely". Consider a bet: you get $1 if the bridge strength is "exactly 100 tons", and pay $1 if the bridge strength is"not 100 tons". Do you think the bet is fair? It seems that no one thinks so. Hence the conclusion (c) is unacceptable because "exactly 100 tons" is almost impossible compared with "not 100 tons". This paradox shows that those indeterminate quantities like the bridge strength cannot be quantified by possibility measure and then they are not fuzzy concepts.

B.8 What is the difference between uncertainty theory and possibility theory?

The essential difference between uncertainty theory (Liu [76]) and possibility theory (Zadeh [190]) is that the former assumes

$$\mathcal{M}\{\Lambda_1 \cup \Lambda_2\} = \mathcal{M}\{\Lambda_1\} \vee \mathcal{M}\{\Lambda_2\} \tag{B.26}$$

only for independent events Λ_1 and Λ_2, and the latter holds

$$\text{Pos}\{\Lambda_1 \cup \Lambda_2\} = \text{Pos}\{\Lambda_1\} \vee \text{Pos}\{\Lambda_2\} \tag{B.27}$$

for any events Λ_1 and Λ_2 no matter if they are independent or not. A lot of surveys showed that the measure of a union of events is usually greater than the maximum of the measures of individual events when they are not independent. This fact states that human brains do not behave fuzziness.

Both uncertainty theory and possibility theory attempt to model belief degrees, where the former uses the tool of uncertain measure and the latter uses the tool of possibility measure. Thus they are complete competitors.

B.9 Why is stochastic differential equation not suitable for modelling stock price?

The origin of stochastic finance theory can be traced to Louis Bachelier's doctoral dissertation *Théorie de la Speculation* in 1900. However, Bachelier's work had little impact for more than a half century. After Kiyosi Ito invented stochastic calculus [55] in 1944 and stochastic differential equation [56] in 1951, stochastic finance theory was well developed among others by Samuelson [130], Black-Scholes [3] and Merton [113] during the 1960s and 1970s.

Traditionally, stochastic finance theory presumes that the stock price (including interest rate and currency exchange rate) follows Ito's stochastic differential equation. Is it really reasonable? In fact, this widely accepted presumption was challenged among others by Liu [88] via some paradoxes.

First Paradox: As an example, let us assume that the stock price X_t follows the differential equation,

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = eX_t + \sigma X_t \cdot \text{“noise”} \tag{B.28}$$

where e is the log-drift, σ is the log-diffusion, and “noise” is a stochastic process. Now we take the mathematical interpretation of the “noise” term as

$$\text{“noise”} = \frac{\mathrm{d}W_t}{\mathrm{d}t} \tag{B.29}$$

where W_t is a Wiener process[3]. Thus the stock price X_t follows the stochastic differential equation,

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = eX_t + \sigma X_t \frac{\mathrm{d}W_t}{\mathrm{d}t}. \tag{B.30}$$

Note that the “noise” term

$$\frac{\mathrm{d}W_t}{\mathrm{d}t} \sim \mathcal{N}\left(0, \frac{1}{\mathrm{d}t}\right) \tag{B.31}$$

is a normal random variable whose expected value is 0 and variance tends to ∞. This setting is very different from other disciplines (e.g. statistics) that usually take

$$\mathcal{N}(0,1) \quad \text{(whose variance is 1 rather than } \infty\text{)} \tag{B.32}$$

as the “noise” term. In addition, since the right-hand part of (B.30) has an infinite variance at any time t, the left-hand part (i.e., the instantaneous growth rate $\mathrm{d}X_t/\mathrm{d}t$ of stock price) has to take an infinite variance at every time. However, the growth rate usually has a finite variance in practice, or at least, it is impossible to have infinite variance at every time. Thus it is impossible that the real stock price X_t follows Ito's stochastic differential equation.

Second Paradox: Roughly speaking, the sample path of a stochastic differential equation (B.30) is increasing with probability 0.5 and decreasing with probability 0.5 at each time no matter what happened before. However, in practice, when the stock price is greatly increasing at the moment, usually it will continue to increase; when the stock price is greatly decreasing, usually

[3] A stochastic process W_t is said to be a Wiener process if (i) $W_0 = 0$ and almost all sample paths are continuous (but non-Lipschitz), (ii) W_t has stationary and independent increments, and (iii) every increment $W_{s+t} - W_s$ is a normal random variable with expected value 0 and variance t.

it will continue to decrease. This means that the stock price in the real world does not behave like Ito's stochastic differential equation.

Third Paradox: It follows from the stochastic differential equation (B.30) that X_t is a geometric Wiener process, i.e.,

$$X_t = X_0 \exp((e - \sigma^2/2)t + \sigma W_t) \tag{B.33}$$

from which we derive

$$W_t = \frac{\ln X_t - \ln X_0 - (e - \sigma^2/2)t}{\sigma} \tag{B.34}$$

whose increment is

$$\Delta W_t = \frac{\ln X_{t+\Delta t} - \ln X_t - (e - \sigma^2/2)\Delta t}{\sigma}. \tag{B.35}$$

Write

$$A = -\frac{(e - \sigma^2/2)\Delta t}{\sigma}. \tag{B.36}$$

Note that the stock price X_t is actually a step function of time with a finite number of jumps although it looks like a curve. During a fixed period (e.g. one week), without loss of generality, we assume that X_t is observed to have 100 jumps. Now we divide the period into 10000 equal intervals. Then we may observe 10000 samples of X_t. It follows from (B.35) that ΔW_t has 10000 samples that consist of 9900 A's and 100 other numbers:

$$\underbrace{A,\ A, \cdots, A,}_{9900}\quad \underbrace{B,\ C, \cdots, Z.}_{100} \tag{B.37}$$

Nobody can believe that those 10000 samples follow a normal probability distribution with expected value 0 and variance Δt. This fact is in contradiction with the property of Wiener process that the increment ΔW_t is a normal random variable. Therefore, the real stock price X_t does not follow the stochastic differential equation.

Perhaps some people think that the stock price does behave like a geometric Wiener process (or Ornstein-Uhlenbeck process) in macroscopy although they recognize the paradox in microscopy. However, as the very core of stochastic finance theory, Ito's calculus is just built on the microscopic structure (i.e., the differential $\mathrm{d}W_t$) of Wiener process rather than macroscopic structure. More precisely, Ito's calculus is dependent on the presumption that $\mathrm{d}W_t$ is a normal random variable with expected value 0 and variance $\mathrm{d}t$. This unreasonable presumption is what causes the second order term in Ito's formula,

$$\mathrm{d}X_t = \frac{\partial h}{\partial t}(t, W_t)\mathrm{d}t + \frac{\partial h}{\partial w}(t, W_t)\mathrm{d}W_t + \frac{1}{2}\frac{\partial^2 h}{\partial w^2}(t, W_t)\mathrm{d}t. \tag{B.38}$$

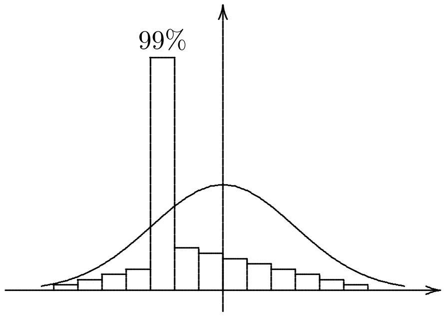


Figure B.2: There does not exist any continuous probability distribution (curve) that can approximate to the frequency (histogram) of ΔW_t. Hence it is impossible that the real stock price X_t follows any Ito's stochastic differential equation.

In fact, the increment of stock price is impossible to follow any continuous probability distribution.

On the basis of the above three paradoxes, personally I do not think Ito's calculus can play the essential tool of finance theory because Ito's stochastic differential equation is impossible to model stock price. As a substitute, uncertain calculus may be a potential mathematical foundation of finance theory. We will have a theory of uncertain finance if the stock price, interest rate and exchange rate are assumed to follow uncertain differential equations.

B.10 In what situations should we use uncertainty theory?

Keep in mind that uncertainty theory is not suitable for modelling frequencies. Personally, I think we should use uncertainty theory in the following five situations.

(i) We should use uncertainty theory (here it refers to uncertain variable) to quantify the future when no samples are available. In this case, we have to invite some domain experts to evaluate the belief degree that each event will happen, and uncertainty theory is just the tool to deal with belief degrees.

(ii) We should use uncertainty theory (here it refers to uncertain variable) to quantify the future when an emergency arises, e.g., war, flood, earthquake, accident, and even rumour. In fact, in this case, all historical data are no longer valid to predict the future. Essentially, this situation equates to (i).

(iii) We should use uncertainty theory (here it refers to uncertain variable) to quantify the past when precise observations or measurements are impossible to perform, e.g., carbon emission, social benefit and oil reserves. In this case, we have to invite some domain experts to estimate them, thus obtaining their uncertainty distributions.

(iv) We should use uncertainty theory (here it refers to uncertain set) to model unsharp concepts, e.g., “young”, “tall”, “warm”, and “most” due to the ambiguity of human language.

(v) We should use uncertainty theory (here it refers to uncertain differential equation) to model dynamic systems with continuous-time noise, e.g., stock price, heat conduction, and vibration.

B.11 How did “uncertainty” evolve over the past 100 years?

After the word “randomness” was used to represent probabilistic phenomena, Knight (1921) and Keynes (1936) started to use the word “uncertainty” to represent any non-probabilistic phenomena. The academic community also calls it Knightian uncertainty, Keynesian uncertainty, or true uncertainty. Unfortunately, it seems impossible for us to develop a mathematical theory to deal with such a broad class of uncertainty because “non-probability” represents too many things. This disadvantage makes uncertainty in the sense of Knight and Keynes not able to become a scientific terminology. Despite that, we have to recognize that they made a great process to break the monopoly of probability theory.

However, there existed two major retrogressions on this issue during that period. The first retrogression arose from Ramsey (1931) with the Dutch book argument that “proves” belief degree follows the laws of probability theory. On the one hand, I strongly disagree with the Dutch book argument. On the other hand, even if we accept the Dutch book argument, we can only prove belief degree meets the normality, nonnegativity and additivity axioms of probability theory, but cannot prove it meets the product probability theorem. In other words, Dutch book argument cannot prove probability theory is able to model belief degree. The second retrogression arose from Cox's theorem (1946) that belief degree is isomorphic to a probability measure. Many people do not notice that Cox's theorem is based on an unreasonable assumption, and then mistakenly believe that uncertainty and probability are synonymous. This idea remains alive today under the name of subjective probability. Yet numerous experiments demonstrated that belief degree does not follow the laws of probability theory.

An influential exploration by Zadeh (1965) was the fuzzy set theory that was widely said to be successfully applied in many areas of our life. However, fuzzy set theory has neither evolved as a mathematical system nor become a suitable tool for rationally modelling belief degrees. The main mistake of fuzzy set theory is based on the wrong assumption that the belief degree of a union of events is the maximum of the belief degrees of the individual events no matter if they are independent or not. A lot of surveys showed that human brains do not behave fuzziness in the sense of Zadeh.

The latest development was uncertainty theory founded by Liu (2007).

Nowadays, uncertainty theory has become a branch of pure mathematics that is not only a formal study of an abstract structure (i.e., uncertainty space) but also applicable to modelling belief degrees. Perhaps some readers may complain that I never clarify what uncertainty is in this book. I think we can answer it this way now. *Uncertainty is anything that follows the laws of uncertainty theory.* From then on, "uncertainty" became a scientific terminology on the basis of uncertainty theory.

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List of Frequently Used Symbols

$\mathcal{M}$	uncertain measure
$(\Gamma, \mathcal{L}, \mathcal{M})$	uncertainty space
ξ, η, τ	uncertain variables
Φ, Ψ, Υ	uncertainty distributions
Φ^{-1}, Ψ^{-1}, Υ^{-1}	inverse uncertainty distributions
μ, ν, λ	membership functions
μ^{-1}, ν^{-1}, λ^{-1}	inverse membership functions
$\mathcal{L}(a, b)$	linear uncertain variable
$\mathcal{Z}(a, b, c)$	zigzag uncertain variable
$\mathcal{N}(e, \sigma)$	normal uncertain variable
$\mathcal{LOGN}(e, \sigma)$	lognormal uncertain variable
(a, b, c)	triangular uncertain set
(a, b, c, d)	trapezoidal uncertain set
E	expected value
V	variance
H	entropy
X_t, Y_t, Z_t	uncertain processes
C_t	Liu process
N_t	renewal process
$\mathcal{Q}$	uncertain quantifier
$(\mathcal{Q}, S, P)$	uncertain proposition
$\forall$	universal quantifier
$\exists$	existential quantifier
$\vee$	maximum operator
$\wedge$	minimum operator
$\neg$	negation symbol
Pr	probability measure
$(\Omega, \mathcal{A}, \Pr)$	probability space
Ch	chance measure
k-max	the kth largest value
k-min	the kth smallest value
$\emptyset$	the empty set
$\Re$	the set of real numbers
iid	independent and identically distributed

Index

Baoding Liu
Uncertainty Theory

When no samples are available to estimate a probability distribution, we have to invite some domain experts to evaluate the belief degree that each event will happen. Perhaps some people think that the belief degree should be modeled by subjective probability or fuzzy set theory. However, it is usually inappropriate because both of them may lead to counterintuitive results in this case. In order to rationally deal with personal belief degrees, uncertainty theory was founded in 2007 and subsequently studied by many researchers. Nowadays, uncertainty theory has become a branch of mathematics.

This is an introductory textbook on uncertainty theory, uncertain programming, uncertain risk analysis, uncertain reliability analysis, uncertain set, uncertain logic, uncertain inference, uncertain process, uncertain calculus, uncertain differential equation, and uncertain statistics. This textbook also shows applications of uncertainty theory to scheduling, logistics, network optimization, data mining, control, and finance.

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\}=1$ for the universal set Γ.

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$ for any event Λ.

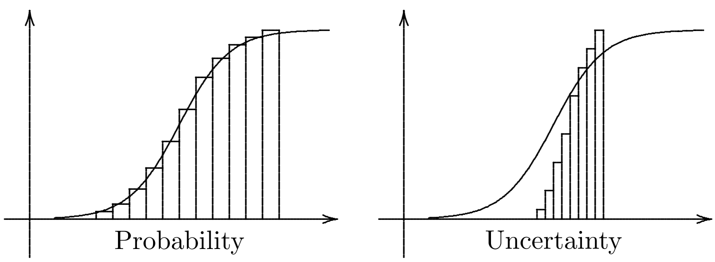
Axiom 3. (Subadditivity Axiom) For every countable sequence of events Λ_1, $\Lambda_2,\cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Axiom 4. (Product Axiom) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k=1,2,\cdots$, respectively.



Probability theory is a branch of mathematics for modelling frequencies, while uncertainty theory is a branch of mathematics for modelling belief degrees.